JOURNAL OF ALGEBRA **184,** 102–158 (1996)
ARTICLE NO. 0251

Hecke Algebras of Type **A** with $q = -1$

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Communicated by Michel Broué

In this paper we study the decomposition matrices of the Hecke algebras of type **A** with $q = -1$ over a field of characteristic 0. We give explicit formulae for the columns of the decomposition matrices indexed by all 2-regular partitions with 1 or 2 parts and an algorithm for calculating the columns of the decomposition matrix indexed by partitions with 3 parts. Combining these results we find all of the rows of the decomposition matrices which are indexed by partitions with at most four parts. All this is accomplished by means of a more general theory which begins by showing that the decomposition numbers in the columns of the decomposition matrices indexed by 2-regular partitions with ‘‘enormous 2-cores’’ are Littlewood–Richardson coefficients.

INTRODUCTION

The Hecke algebras $\mathscr{H}$ of type **A** are q-analogues of the group algebras of the symmetric groups, and a complete knowledge of the structure of the Hecke algebras would give the decomposition matrices of the symmetric groups, and the decomposition matrices of the general linear groups in all characteristics [4]. We study, in this paper, one of the simplest cases where $\mathscr{H}$ is not semisimple; namely, we take $q = -1$, and we assume that the underlying field has characteristic 0. The information which we glean about the decomposition matrix D of $\mathscr{H}$ then has implications for the 2-modular decomposition matrices of the symmetric groups, since D is a ‘‘first approximation’’ to the 2-modular decomposition matrix of the symmetric group. To be precise, the latter matrix is obtained from D by

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0021-8693/96 $18.00

postmultiplying by a lower unitriangular matrix, all of whose entries are non-negative integers (see Geck [5, Theorem 5.3] and the conjecture of the first author in [8, Sect. 4]).

In comparison with the current state of knowledge for the symmetric group, we are able to make considerable progress. We give an easy construction of the first column of the decomposition matrix of $\mathscr{H}$, when $q = -1$ (see Theorem 3.2); no method is known for determining the first or any other column of the 2-modular decomposition matrix of the symmetric group. We also give explicit formulae for the columns of the decomposition matrix of $\mathscr{H}$ which are indexed by 2-part partitions (see Section 7) and an algorithm for calculating those columns corresponding to 3-part partitions (with one ambiguity; see Theorem 8.1). For the symmetric groups, a similar target seems far out of reach. In the symmetric group case the rows of the decomposition matrix corresponding to 2-part partitions are known [7, Chap. 24], as are those labelled by partitions of the form $(a, b, 1)$ [6, 13]; no other rows are known. We determine all of the rows of the decomposition matrix of $\mathscr{H}$ indexed by partitions with at most 4 parts (see Corollary 7.2, Corollary 8.21, and Corollary 8.22).

One reason why we are able to make such progress in calculating the decomposition numbers of $\mathscr{H}$ is that, relative to the symmetric group case, a large number of the Specht modules for $\mathscr{H}$ are irreducible. This allows us to compute the rows of the decomposition matrix which are indexed by partitions whose 2-weight is small compared to the size of their 2-core. For such rows we show that the decomposition numbers are Littlewood−Richardson coefficients. We then use these results to describe certain columns of the decomposition matrix and in turn deduce some information about arbitrary columns of D. In particular, we obtain a result, which we call the "Scattering Theorem" (6.4), which gives a combinatorial rule for calculating certain decomposition numbers. This result allows us to classify the irreducible Specht modules in $\mathscr{H}$ and it is the key to calculating the columns of the decomposition matrix indexed by 3-part partitions.

Throughout, our methods are primarily combinatorial; in particular, the Littlewood−Richardson rule plays a crucial role. In part this is because we concentrate upon the projective indecomposable modules for $\mathscr{H}$ and study their "α-restrictions" and "α-inductions," where $\alpha = 0$ or 1. In this way we are able to start with a known indecomposable and from it construct new projective $\mathscr{H}$-modules which we then analyse using various combinatorial methods from the general theory as developed and described in [8].

Finally, it should be mentioned that throughout this research we benefited greatly from calculations made using GAP [12]. In this way we were able to look in detail at examples for large partitions which would otherwise have remained out of reach.

1. PRELIMINARIES

Throughout this paper n is a non-negative integer, $\mathfrak{S}_n$ is the symmetric group of degree n, and $\mathbb{F}$ is a field of characteristic 0. For each non-zero $q \in \mathbb{F}$ the Hecke algebra $\mathscr{H}_{\mathbb{F},n,q}$ is the associative unital $\mathbb{F}$-algebra with basis elements T_w indexed by elements w of $\mathfrak{S}_n$, the multiplication on $\mathscr{H}_{\mathbb{F},n,q}$ being defined as follows. For $w \in \mathfrak{S}_n$ and s a basic transposition,

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) = l(w) + 1 \\ qT_{ws} + (q-1)T_w & \text{if } l(w) = l(w) - 1, \end{cases}$$

where l is the usual length function on $\mathfrak{S}_n$.

Dipper and James [2, 3] have shown that the representation theory of $\mathscr{H}_{\mathbb{F},n,q}$ is to a large extent dependent upon the integer e which is the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$. For example, the simple $\mathscr{H}_{\mathbb{F},n,q}$-modules are indexed by e-regular partitions and the blocks of $\mathscr{H}_{\mathbb{F},n,q}$ depend upon the e-cores. Almost exclusively, we shall be concerned with the algebra $\mathscr{H}_{\mathbb{F},n,-1}$ and throughout we use the abbreviation $\mathscr{H}_n$, or simply $\mathscr{H}$, for $\mathscr{H}_{\mathbb{F},n,-1}$. In our case $1 + q = 0$, so $e = 2$ and we deal with 2-regular partitions, 2-cores, and so on.

For each partition λ of n there exists a Specht module $\mathscr{S}(\lambda)$ [2]; this is an $\mathscr{H}$-module whose dimension is equal to the number of standard λ-tableaux. We define $\mathscr{S}(\lambda)$ to be 0 if λ is not a partition.

Let $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ denote the partition which is conjugate to the partition λ. Note that $\mathscr{S}(\lambda) = \mathscr{S}(\lambda')$ since $q = -1$.

If all of the parts of λ are distinct then λ is 2-*regular*; otherwise λ is 2-*singular*. In the case where λ is 2-regular, $\mathscr{S}(\lambda)$ has a maximal $\mathscr{H}$-submodule $\mathscr{S}(\lambda)^{\max}$ [2], and its simple quotient $\mathscr{S}(\lambda)/\mathscr{S}(\lambda)^{\max}$ is denoted by $\mathscr{D}(\lambda)$. Moreover, as λ runs over the 2-regular partitions of n, $\mathscr{D}(\lambda)$ runs over a complete set of non-isomorphic simple $\mathscr{H}$-modules.

Given two partitions λ and μ of n, where μ is 2-regular, denote by $d_{\lambda\mu}$ the composition multiplicity of $\mathscr{D}(\mu)$ as a factor of $\mathscr{S}(\lambda)$. The non-negative integers $d_{\lambda\mu}$ are the *decomposition numbers* of $\mathscr{H}$ and they are the object of our study. We next recall several properties of the decomposition numbers.

(1.1) THEOREM (Dipper and James [2, Theorem 7.6]). *Suppose that λ and μ are partitions of n where μ is 2-regular. Then $d_{\lambda\mu} = 0$ unless $\lambda \trianglelefteq \mu$ and $d_{\mu\mu} = 1$.*

Here, the symbol "$\trianglelefteq$" denotes the usual dominance ordering of partitions [7, Definition 3.2]. Often the decomposition numbers for the Hecke algebra $\mathscr{H}_{\mathbb{F},n,q}$ can be found from the decomposition numbers of $\mathscr{H}_{\mathbb{F},m,q}$

where (typically, but not always) $m < n$. In particular, the next result is quite useful.

(1.2) THEOREM (Row and Column Removal, James [8, Rule 5.8]). *Let* $\lambda = (\lambda_1, \dots, \lambda_r)$ *and* $\mu = (\mu_1, \dots, \mu_s)$ *be two partitions of* n *where* μ *is* 2-*regular*.

(i) *Suppose that* $\lambda_1 = \mu_1$. *Then* $d_{\lambda\mu} = d_{(\lambda_2, \dots, \lambda_r)(\mu_2, \dots, \mu_s)}$.

(ii) *Suppose that* $r = s$ (*i.e.*, $\lambda'_1 = \mu'_1$). *Then* $d_{\lambda\mu} = d_{(\lambda_1 - 1, \dots, \lambda_r - 1)(\mu_1 - 1, \dots, \mu_r - 1)}$.

There is a more general result of the first author [8, Theorem 6.18], which allows the removal of two or more rows or columns from a partition; however, as we do not use it often we shall refer to [8] as needed.

To a large extent we work with the projective indecomposable $\mathscr{H}$-modules. As usual, there is a projective indecomposable $\mathscr{H}$-module $\mathscr{P}(\mu)$ for each 2-regular partition μ of n. Moreover, $\mathscr{P}(\mu)$ has the same composition factors as $\sum_\lambda d_{\lambda\mu}\mathscr{S}(\lambda)$. We abuse notation and write $\mathscr{P}(\mu) = \sum_\lambda d_{\lambda\mu}\mathscr{S}(\lambda)$. In addition, if $d_{\lambda\mu} > 0$ then we sometimes write $\mathscr{S}(\lambda) \subseteq \mathscr{P}(\mu)$.

Note in particular that if P is any projective $\mathscr{H}$-module then it is a direct sum of indecomposable $\mathscr{H}$-modules and consequently it has the same composition factors as a sum of Specht modules. Further, if all of the Specht modules in P are known then in order to write P as a direct sum of indecomposable $\mathscr{H}$-modules it is only necessary to consider those 2-regular partitions μ for which $\mathscr{S}(\mu) \subseteq P$.

If M is an $\mathscr{H}_n$-module then $M\downarrow$ denotes M regarded as an $\mathscr{H}_{n-1}$-module and $M\uparrow$ denotes the $\mathscr{H}_{n+1}$ module induced from M. For the Specht module $\mathscr{S}(\lambda)$, both $\mathscr{S}(\lambda)\downarrow$ and $\mathscr{S}(\lambda)\uparrow$ have filtrations by Specht modules, the factors being described by the Branching Theorem [7, Chap. 9]. For example,

$$\mathscr{S}(3,2)\downarrow = \mathscr{S}(2,2) + \mathscr{S}(3,1),$$

and

$$\mathscr{S}(3,2)\uparrow = \mathscr{S}(4,2) + \mathscr{S}(3,3) + \mathscr{S}(3,2,1).$$

Since $\mathscr{P}(\mu) = \sum_\lambda d_{\lambda\mu}\mathscr{S}(\lambda)$ this allows us to calculate $\mathscr{P}(\mu)\downarrow$, which is a projective $\mathscr{H}_{n-1}$-module, and $\mathscr{P}(\mu)\uparrow$, a projective $\mathscr{H}_{n+1}$-module.

More generally, if $\lambda \vdash n$ and $\mu \vdash m$ then $\mathscr{S}(\lambda) \otimes \mathscr{S}(\mu)$ is an $\mathscr{H}_n \times \mathscr{H}_m$ module, and we can consider the induced $\mathscr{H}_{n+m}$-module $\mathscr{S}(\lambda) \otimes \mathscr{S}(\mu)\uparrow$, which we denote by $\mathscr{S}(\lambda) \circ \mathscr{S}(\mu)$. Then $\mathscr{S}(\lambda) \circ \mathscr{S}(\mu)$ has a filtration by Specht modules and the factors are given by the Littlewood–Richardson rule [7, Chap. 16]; explicitly,

$$\mathscr{S}(\lambda)\circ\mathscr{S}(\mu) = \sum_{\nu\vdash n+m} \alpha^{\nu}_{\lambda\mu}\mathscr{S}(\nu),$$

where $\alpha^{\nu}_{\lambda}\mu$ denotes the Littlewood–Richardson coefficient. Similarly we define $\mathscr{P}(\lambda)\circ\mathscr{P}(\mu)$ for 2-regular partitions λ and μ.

The next result is a very powerful aid to our calculations.

(1.3) THEOREM (Dipper and James [3, Cor. 4.4]). *Suppose that λ and μ are partitions of n where μ is 2-regular. Then $d_{\lambda\mu} = 0$ unless λ and μ have the same 2-core.*

Recall that the 2-*core* $\tilde{\lambda}$ of a partition λ is what remains after as many as possible skew 2-hooks have been successively removed from the diagram of λ; the 2-*weight* of λ is the number of skew 2-hooks which have to be taken off to reach the 2-core. The 2-cores are easy to describe; they are the "triangular" partitions of the form $(k, k-1, \ldots, 3, 2, 1)$ for some $k \geq 0$.

One way to determine when two partitions have the same 2-core is to look at their 2-residues. The 2-*residue* of the node in row i and column j of the diagram $[\lambda]$ of the partition λ is 0 if $j-i$ is even and 1 otherwise. Then two partitions of n have the same 2-core if and only if their diagrams have the same number of nodes with 2-residue 0 (this is easy to see since every skew 2-hook contains exactly one node with 2-residue 0).

Let $\alpha = 0$ or 1. We define $\mathscr{S}(\lambda)\downarrow\alpha$ to be the sum of those $\mathscr{S}(\nu)$ where ν is obtained from λ be removing a node of 2-residue α (using the Branching Theorem as above). Since $\mathscr{P}(\mu) = \sum_{\lambda} d_{\lambda\mu}\mathscr{S}(\lambda)$ we define $\mathscr{P}(\mu)\downarrow\alpha = \sum_{\lambda} d_{\lambda\mu}\mathscr{S}(\lambda)\downarrow\alpha$. Theorem 1.3 shows that $\mathscr{P}(\mu)\downarrow\alpha$ is a projective module (or, to be accurate, $\mathscr{P}(\mu)\downarrow\alpha$ gives the Specht modules which appear in a projective module). In a similar way, we define $\mathscr{S}(\lambda)\uparrow\alpha$ and $\mathscr{P}(\mu)\uparrow\alpha$. Abbreviations such as $\mathscr{P}(\mu)\uparrow 0^2 10$ are to be interpreted to mean $\mathscr{P}(\mu)\uparrow 0\uparrow 0\uparrow 1\uparrow 0$.

The two procedures described in the last paragraph are called *α-restriction* and *α-induction* respectively. Many examples, in the context of the symmetric groups, are given in [9, Chap. 6].

One of the reasons why we are able to make considerable progress with Hecke algebras with $q = -1$ is that there are many 2-regular partitions λ for which $\mathscr{S}(\lambda) = \mathscr{D}(\lambda)$. Next we give a sufficient condition for the Specht module $\mathscr{S}(\lambda)$ to be irreducible; in (6.5) below we shall show that this condition is also necessary (recently, in [10], we were able to classify the irreducible e-regular Specht modules of arbitrary Hecke algebras $\mathscr{H}_{\mathbb{F},n,q}$).

(1.4) THEOREM. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_r > 0$, is a partition such that $\lambda_i - \lambda_{i+1}$ is odd for $1 \leq i \leq r-1$. Then $\mathscr{S}(\lambda) = \mathscr{D}(\lambda)$.*

Proof. For the moment, let $\mathbb{F} = \mathbb{Q}(q)$ where q is transcendental. Then Dipper and James [3, Theorem 4.11] have given a formula for the determinant of the Gram matrix, with respect to the standard basis of the Specht module $\mathscr{S}(\lambda)$ for $\mathscr{H}_{\mathbb{F},n,q}$. This determinant is of the form $q^m f(q)$ for some integer m and some polynomial $f(q) \in \mathbb{Q}[q]$. Starting with this formula, the technique used by James and Murphy in [11] can be applied to rewrite $f(q)$ as a product of terms of the form $[h_{ac}]_q/[h_{bc}]_q$ where h_{ij} denotes the length of the (i, j)-hook in λ and $[r]_q = 1 + q + \cdots + q^{r-1}$ for all integers r.

Assume now that every $\lambda_i - \lambda_{i+1}$ is odd. Then, for all a, b, and c the integers h_{ac} and h_{bc} have the same parity. Hence, $[h_{ac}]_q/[h_{bc}]_q$ can be expressed as a quotient of two polynomials, neither of which is divisible by $1 + q$. Therefore, $1 + q$ does not divided $f(q)$, and so $f(-1) \neq 0$. It follows that the Gram matrix in our case, where $q = -1$, is non-singular, and thus $\mathscr{S}(\lambda) = \mathscr{D}(\lambda)$. ∎

Partitions satisfying the conditions of the theorem play an important part in what follows so we make the definition:

(1.5) DEFINITION. A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_r > 0$, is *alternating* if $\lambda_i - \lambda_{i+1}$ is odd for all i where $1 \leq i \leq r - 1$.

In outline, our method for finding decomposition numbers is as follows. First, we exploit Theorem 1.4 in order to find the projective indecomposable $\mathscr{H}$-modules $\mathscr{P}(\lambda)$ in the case where the 2-weight of λ is small compared to the size of the 2-core of λ. Once we have a supply of projective indecomposable modules we α-restrict and α-induce them to produce projective modules P which may or may not be indecomposable. Then we apply various combinatorial tricks, such as Theorem 1.2, to determine whether or not known indecomposables are contained in P, and if they are then we "subtract" them. After we have removed as much as possible from P, we try to prove that what remains is indecomposable.

2. PARTITIONS WITH ENORMOUS 2-CORES

In this section we give an explicit description of some of the rows and columns of the decomposition matrix of $\mathscr{H}$ which correspond to partitions which have an "enormous 2-core." These results are fundamental to the rest of the paper.

In order to appreciate what it means for a partition to have an enormous 2-core we first look more closely at the process of removing 2-hooks from a partition in order to reach its 2-core. For convenience we identify the *diagram* of λ with the array of points

$$[\lambda] = \{(i, j) : 1 \leq i \leq l(\lambda) \text{ and } 1 \leq j \leq \lambda_i\} \subseteq \mathbb{Z} \times \mathbb{Z},$$

where $l(\lambda)$ is the length of λ (i.e., the number of non-zero parts in λ). A *domino* is a two element subset of $[\lambda]$ of the form $\{(i, j), (i + 1, j)\}$ or

$\{(i,j),(i,j+1)\}$. Dominoes of the first kind are *horizontal* dominoes and those of the second kind are *vertical*. A domino $\mathfrak{d}$ in (the diagram of) λ is *removable* if there exists a sequence of disjoint dominoes $\mathfrak{d}_0,\dots,\mathfrak{d}_k=\mathfrak{d}$ such that $\lambda-\mathfrak{d}_0-\cdots-\mathfrak{d}_i$ is the diagram of a partition for $i=0,\dots,k$. Two dominoes in λ *intersect* if they have at least one node in common.

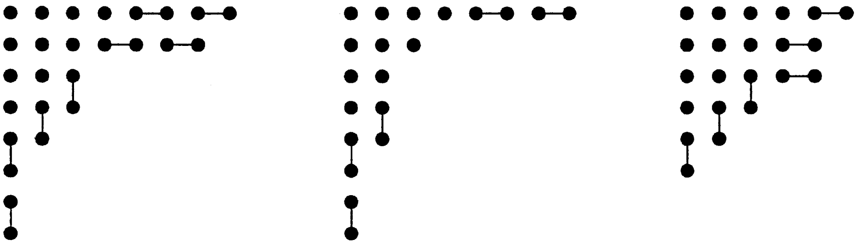
(2.1) DEFINITION. We say that a partition λ is 2-*quotient separated* if whenever $\mathfrak{d}$ is a removable domino in λ then either $\mathfrak{d}$ is horizontal and all removable vertical dominoes in λ appear in rows strictly below $\mathfrak{d}$, or $\mathfrak{d}$ is vertical and all removable horizontal dominoes appear in columns strictly to the right of $\mathfrak{d}$.

If λ is a 2-quotient separated partition then the removable dominoes in $[\lambda]$ occur either as horizontal dominoes at the "top" of λ or as vertical dominoes at the "bottom" of λ. Consequently, we write $\lambda=(\lambda^h;\lambda^v)_c$ where λ^h and λ^v are partitions such that λ_i^h is the number of removable dominoes in the ith row of λ and λ_j^v is the number of removable vertical dominoes in the jth column of λ, and $c=l(\tilde{\lambda})$ (note that the 2-cores are uniquely determined by their length). For "enormous 2-cores" the results of this section are essentially independent of the 2-core so we will often omit the subscript c from this notation.

Note that the components of the 2-quotient of λ are λ^h and $(\lambda^v)'$. Consequently a 2-quotient separated partition λ is alternating if and only if $\lambda^v=(0)$.

(2.2) DEFINITION. A partition has an *enormous* 2-*core* if the 2-weight of λ is less than or equal to $l(\tilde{\lambda})+1$.

(2.3) EXAMPLE. Consider the three partitions $(8,7,3^2,2,1^3)=(2^2;2,1^2)_4$, $(8,3,2^3,1^3)=(2;2,1)_4$, $(6,5^2,3,2,1)$, and their diagrams below. It is easy to see that just the first two of these partitions are 2-quotient separated and that only the second partition has an enormous 2-core.



(2.4) LEMMA. *Suppose that λ has an enormous 2-core. Then λ is 2-quotient separated.*

Proof. If λ is a partition with $c = l(\tilde{\lambda})$ then λ is not 2-quotient separated if and only if $(i, c+3-i) \in [\lambda]$ for $1 \le i \le c+2$. This requires also that $(i, c+2-i) \in [\lambda]$ for $1 \le i \le c+1$. Thus the number of removable dominoes in $[\lambda]$ is at least $c+2$, and so the 2-weight of λ is at least $c+2$. ∎

Given a partition λ with an enormous 2-core we may write $\lambda = (\lambda^h; \lambda^v)_c$.

We can now state the main result of this section. It should be noted that the only property of the Hecke algebra $\mathscr{H}$ needed in the proof is that $\mathscr{S}(\lambda) = \mathscr{S}(\lambda')$ for all partitions λ. Consequently the theorem also holds for any Hecke algebra where $q+1=0$, and in particular for the symmetric group over fields of characteristic 2.

(2.5) THEOREM. *Suppose that $\lambda = (\lambda^h; \lambda^v)_c$ has an enormous 2-core and let $h = |\lambda^h|$ and $v = |\lambda^v|$. Then*

$$\mathscr{S}(\lambda) = \sum_{\nu \vdash h+v} \alpha^{\nu}_{\lambda^n \lambda^v} \mathscr{S}\big((\nu; (0))_c\big),$$

where $\alpha^{\nu}_{\lambda^h\lambda^v}$ is the Littlewood–Richardson coefficient which gives the multiplicity of $\mathscr{S}(\nu)$ in $\mathscr{S}(\lambda^h) \circ \mathscr{S}(\lambda^v)$.

Now partitions of the form $(\nu; (0))_c$ are alternating, so $\mathscr{S}((\nu; (0)_c) = \mathscr{D}((\nu; (0))_c)$ by Theorem 1.4. Therefore, Theorem 2.5 gives the rows of the decomposition matrix of $\mathscr{H}$ which correspond to 2-singular partitions which have enormous 2-cores. Explicitly,

(2.6) COROLLARY. *Suppose that $\lambda = (\lambda^h; \lambda^v)_c$ has an enormous 2-core. Then for any 2-regular partition μ,*

$$d_{\lambda\mu} = \begin{cases} \alpha^{\nu}_{\lambda^h\lambda^v} & \text{if } \mu = (\nu; (0))_c \\ 0 & \text{otherwise}. \end{cases}$$

To prove the theorem we need to introduce some notation. Given an integer m define the virtual module $\omega(m)$ by

$$\omega(m) = \sum_{j=0}^{m-1} (-1)^j \mathscr{S}(m-j, 1^j).$$

Note $\omega(m) = 0$ if m is even since $\mathscr{S}(x, 1^y) = \mathscr{S}(y+1, 1^{x-1})$ for all $x, y \ge 1$. It is known, see [7, Remark, p. 84], that $\mathscr{S}(\lambda) \circ \omega(m)$ is the linear combination of Specht modules whose partitions are obtained from λ by "wrapping skew m-hooks" onto λ in all possible ways and attaching a sign of $(-1)^l$ where l is the leg length of the skew hook.

For an arbitrary partition $\mu = (\mu_1, \ldots, \mu_s)$ let

$$\omega(\mu) = \omega(\mu_1) \circ \cdots \circ \omega(\mu_s)$$

and define coefficients $\omega_{\lambda\mu\nu}$ by

$$\mathscr{S}(\lambda) \circ \omega(\mu) = \sum_{\nu} \omega_{\lambda\mu\nu} \mathscr{S}(\nu).$$

A reformulation of the Murnaghan−Nakayama Rule reveals that some of these coefficients are known. For partitions τ and ν of n let χ^τ denote the irreducible complex character of $\mathfrak{S}_n$ associated with τ and let $\chi^\tau(\nu)$ denote the value χ^τ takes upon the ν-conjugacy class of $\mathfrak{S}_n$. Then we have the following result.

(2.7) Lemma (Murnaghan−Nakayama Rule [7, Theorem 21.1]). *Suppose that τ and ν are partitions of n. Then $\omega_{(0)\tau\nu} = \chi^\tau(\nu)$.*

The next result is quite technical and is the heart of the proof of Theorem 2.5 (Lemma 2.7 is also crucial).

(2.8) Lemma. *Suppose that λ has an enormous 2-core and write $\lambda = (\lambda^h; \lambda^v)$. Let $h = |\lambda^h|$, $v = |\lambda^v|$, and suppose that $0 \le m \le v$. Then for any partition μ of $v - m$*

$$\sum_{\nu \vdash v} \omega_{\mu(m)\nu} \mathscr{S}(\lambda^h; \nu) = \sum_{\eta \vdash h+m} \omega_{\lambda^h(m)\eta} \mathscr{S}(\eta; \mu).$$

Proof. By the above remarks $\omega(2m) = 0$ so it suffices to show that

$$\text{(2.9)} \qquad \mathscr{S}(\lambda^h; \mu) \circ \omega(2m) = \sum_{\eta \vdash h+m} \omega_{\lambda^h(m)\eta} \mathscr{S}(\eta; \mu) - \sum_{\nu \vdash v} \omega_{\mu(m)\nu} \mathscr{S}(\lambda^h; \nu).$$

It was also stated above that $\mathscr{S}(\lambda^h; \mu) \circ \omega(2m)$ is the linear combination of Specht modules which correspond to partitions σ obtained by wrapping skew $2m$-hooks onto the partition $(\lambda^h; \mu)$ in all possible ways and attaching the appropriate signs. Suppose that σ is such a partition. Then σ has the same 2-core and the same 2-weight as $\lambda = (\lambda^h; \lambda^v)$ and so it also has an enormous 2-core. Therefore, σ is 2-quotient separated, so we may write $\sigma = (\sigma^h; \sigma^v)$, and it follows that when the skew $2m$-hook was added to $(\lambda^h; \mu)$ to form σ either all of the dominoes making up the skew hook were placed horizontally or they were all placed vertically. Equivalently, $\sigma = (\eta; \mu)$ or $\sigma = (\lambda^h; \nu)$ where $\eta \vdash h + m$ and $\nu \vdash v$ respectively.

First consider the case where $\sigma = (\eta; \mu)$ for some η. Then η was obtained by wrapping a $2m$-hook onto the "top" of $(\lambda^h; \mu)$. To the right of its 2-core, $(\lambda^h; \mu)$ has $2\lambda_i^h$ nodes in row i and σ has $2\eta_i$ nodes; so $(\eta_i - \lambda_i^h)$ horizontal dominoes have been added to the ith row of $(\lambda^h; \mu)$ for each i to make η. It follows that $\omega_{(\lambda^h;\,\mu)(2m)\sigma} = \omega_{\lambda^h(m)\eta}$ which establishes the first half of (2.9).

The case where $\sigma = (\lambda^h; \nu)$ is similar, the only difference being that everything is phrased in terms of the conjugate partitions; this introduces a sign change due to a change in the parity of the leg length of the added hook. ∎

We are now in a position to prove Theorem 2.5; however, before we do so we give an example to illustrate the proof.

(2.10) Example. The partition $\lambda = (6, 3, 2^3, 1^3) = (1; 2, 1)_4$ has an enormous 2-core. In order to prove the result for λ we use (2.9) to calculate $(1; (0))_4 \circ \omega(2\mu)$ where μ runs over the partitions (3), (2, 1), and (1^3) of 3:

$$\begin{aligned}
\mathscr{S}(1;3) - \mathscr{S}(1;2,1) + \mathscr{S}(1;1^3) &= \mathscr{S}(4;0) - \mathscr{S}(2^2;0) + \mathscr{S}(1^4;0)\\
-\mathscr{S}(1;3) \qquad\qquad\qquad + \mathscr{S}(1;1^3) &= \mathscr{S}(4;0) + \mathscr{S}(3,1;0) - \mathscr{S}(2,1^2;0)\\
&\quad - \mathscr{S}(1^4;0) - \mathscr{S}(3;1) - \mathscr{S}(1^3;1)\\
&\quad + \mathscr{S}(2;1^2) - \mathscr{S}(1^2;1^2)\\
&\quad - \mathscr{S}(2;2) + \mathscr{S}(1^2;2)\\
\mathscr{S}(1;3) + 2\mathscr{S}(1;2,1) + \mathscr{S}(1;1^3) &= \mathscr{S}(4;0) + 2\mathscr{S}(3,1;0) + 2\mathscr{S}(2^2;0)\\
&\quad + 3\mathscr{S}(2,1^2;0) + \mathscr{S}(1^4;0)\\
&\quad - 3\mathscr{S}(3;1) - 6\mathscr{S}(2,1;1)\\
&\quad - 3\mathscr{S}(1^3;1) + 3\mathscr{S}(2;2)\\
&\quad + 3\mathscr{S}(1^2;2) + 3\mathscr{S}(1^2;1^2) + \mathscr{S}(2;1^2)
\end{aligned}$$

By induction, everything upon the right-hand side is known and hence is expressible as a linear combination of $\mathscr{S}(\eta; (0))$ for $\eta \vdash 4$. As predicted by Lemma 2.7 the coefficients on the left-hand side are from the character table of $\mathfrak{S}_3$. Hence there is a unique way of writing the terms on the left-hand side as a linear combination of the $\mathscr{S}(\eta; (0))$.

Proof of Theorem 2.5. Let $v = |\lambda^v|$ and $h = |\lambda^h|$. We use induction on v. First consider the case where $\lambda^v = (1)$. Taking $\mu = (0)$ in Lemma 2.8

shows that

$$\mathscr{S}(\lambda) = \mathscr{S}(\lambda^h;(1)) = \sum_{\eta \vdash h+1} \omega_{\lambda^h(1)\eta} \mathscr{S}(\eta;(0)).$$

Since it is clear that $\omega_{\sigma(1)\tau} = \alpha^{\tau}_{\sigma(1)}$ the result follows in this case.

Now suppose that $v > 1$ and that the result has been established for all partitions $\sigma = (\sigma^h; \sigma^v)$ with enormous 2-cores and $|\sigma^v| < v$. For any partition $\tau = (\tau_1, \dots, \tau_k)$ of v we wish to consider the product $\mathscr{S}(\lambda^h;(0)) \circ \omega(2\tau) = 0$ where $2\tau = (2\tau_1, \dots, 2\tau_k)$. If $\tau = (v)$ then by Lemma 2.8

$$\sum_{\nu \vdash v} \omega_{(0)(v)\nu} \mathscr{S}(\lambda^h;\nu) = \sum_{\eta \vdash h+v} \omega_{\lambda^h(v)\eta} \mathscr{S}(\eta;(0));$$

so we have written a linear combination of Specht modules $\mathscr{S}(\lambda^h;\nu)$ as a linear combination of Specht modules of the form $\mathscr{S}(\eta;(0))$ where $\nu \vdash v$ and $\eta \vdash h + v$. If τ has more than one part then again by Lemma 2.8

$$\begin{aligned}\sum_{\nu \vdash v} \omega_{(0)(\tau_1)\nu} \mathscr{S}(\lambda^h;\nu) \circ \omega(2\tau_2, \dots, 2\tau_k) \\ = \sum_{\eta \vdash h+v} \omega_{\lambda^h(\tau_1)\eta} \mathscr{S}(\eta;(0)) \circ \omega(2\tau_2, \dots, 2\tau_k).\end{aligned}$$

By applying Lemma 2.8 $(k-1)$ more times and using induction to rewrite any terms of the form $\mathscr{S}(\eta;\nu)$ where $|\nu| < v$, we obtain an equation of the form

$$(2.11)_\tau \qquad \sum_{\nu \vdash v} \omega_{(0)\tau\nu} \mathscr{S}(\lambda^h;\nu) = \sum_{\eta \vdash h+v} \tilde{\omega}_{\lambda^h\tau\eta} \mathscr{S}(\eta;(0))$$

for some coefficients $\tilde{\omega}_{\lambda^h\tau\eta} \in \mathbb{Z}$. So we have expressed $\sum_{\nu \vdash v} \omega_{(0)\tau\nu} \mathscr{S}(\lambda^h;\nu)$ as a linear combination of Specht modules of the form $\mathscr{S}(\eta;(0))$ for $\eta \vdash h + v$.

Now, by Lemma 2.7, $\omega_{(0)\tau\nu} = \chi^{\tau}(\nu)$; so the left-hand side of each equation in $(2.11)_\tau$ is $\sum_\nu \chi^\tau(\nu) \mathscr{S}(\lambda^h;\nu)$. Therefore, since the character table of the symmetric group $\mathfrak{S}_v$ is non-singular, the system of equations in $(2.11)_\tau$ has a unique solution. That is to say, for all partitions ν of v there is a unique way of writing $\mathscr{S}(\lambda^h;\nu)$ as a linear combination of Specht modules of the form $\mathscr{S}(\eta;(0))$ where $\eta \vdash h + v$.

To complete the proof it only remains to note that Eq. (2.9) agrees with the answer given by the Littlewood–Richardson rule; consequently each equation in $(2.11)_\tau$ is also consistent with the Littlewood–Richardson rule. ∎

If λ is a partition with an enormous 2-core then Corollary 2.6 gives a nice answer for the decomposition of the Specht module $\mathscr{S}(\lambda)$ as a sum of simple $\mathscr{H}$-modules $\mathscr{D}(\mu)$. We now use Corollary 2.6 to calculate the columns of the decomposition matrix corresponding to 2-regular partitions with enormous 2-cores. In Theorem 5.7 below we extend this result to a class of 2-regular partitions which are 2-quotient separated but which do not necessarily have enormous 2-cores.

(2.12) THEOREM. *Suppose that* $\mu = (\mu^h; (0))_c$ *is a* 2*-regular partition with an enormous* 2*-core. Then* μ *is the only* 2*-regular partition contained in* $\mathscr{P}(\mu)$ *and for all* λ

$$d_{\lambda\mu} = \begin{cases} \alpha^{\mu^h}_{\lambda^h\lambda^v} & \text{if } \lambda = (\lambda^h; \lambda^v)_c \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Suppose that $\mathscr{S}(\lambda) \subseteq \mathscr{P}(\mu)$. Then λ has the same 2-core and the same 2-weight as μ so it also have an enormous 2-core. Consequently we may write $\lambda = (\lambda^h; \lambda^v)_c$ and moreover $d_{\lambda\mu} = \alpha^{\mu^h}_{\lambda^h\lambda^v}$ by Corollary 2.6.

It remains to show that the only 2-regular partition in $\mathscr{P}(\mu)$ is μ itself. Suppose that λ is 2-regular; then $\lambda^v = (0)$. Therefore, $d_{\lambda\mu} \neq 0$ if and only if $\alpha^{\mu^h}_{\lambda^h(0)} \neq 0$, which is if and only if $\lambda = \mu$. ∎

3. THE FIRST TWO COLUMNS

In this section we give explicit formulae for the columns of the decomposition matrix of $\mathscr{H}$ indexed by the partitions (n) and $(n-1,1)$.

(3.1) LEMMA. *Suppose that* $x \geq 2$ *and* $y \geq 1$. *Then*

(i) $$\mathscr{S}(x, 1^y) = \sum_{\substack{k=0\\2|k}}^{y} \mathscr{S}(x+k, y-k).$$

(ii) $$\mathscr{S}(x, 2, 1^{y-1}) = \sum_{\substack{k=0\\2|k}}^{y} \mathscr{S}(x+k, y-k, 1) + c\mathscr{S}(x+y-1, 2),$$

where $c = 0$ *if* y *is even and* $c = 1$ *otherwise*.

Proof. We may assume that $x \geq y$ since $\mathscr{S}(x, 1^y) = \mathscr{S}(y+1, 1^{x-1})$ and $\mathscr{S}(x, 2, 1^y) = \mathscr{S}(y+2, 2, 1^{x-2})$ for (i) and (ii) respectively.

Now $\mathscr{S}(y) = \mathscr{S}(1^y)$, so by the Littlewood–Richardson rule

$$\begin{aligned} 0 &= \mathscr{S}(x) \circ (\mathscr{S}(y) - \mathscr{S}(1^y)) \\ &= \mathscr{S}(x+y) + \mathscr{S}(x+y-1, 1) + \cdots \\ &\quad + \mathscr{S}(x, y) - \mathscr{S}(x+1, 1^{y-1}) - \mathscr{S}(x, 1^y); \end{aligned}$$

from which (i) follows by induction.

To prove (ii), induce the formulae from (i) from the Hecke algebra $\mathscr{H}_n$ to $\mathscr{H}_{n+1}$ to obtain:

$$\begin{aligned}\mathscr{S}(x+1,1^y) &+\mathscr{S}(x,2,1^{y-1}) +\mathscr{S}(x,1^{y+1})\\ &= \sum_{\substack{k=0\\ 2\mid k}}^{y} \big(\mathscr{S}(x+k+1,y-k) +\mathscr{S}(x+k,y+1-k)\\ &\qquad\qquad +\mathscr{S}(x+k,y-k,1)\big).\end{aligned}$$

If y is odd then $\mathscr{S}(x+1,1^y)$ and $\mathscr{S}(x,1^{y+1})$ can be subtracted from both sides of this equation using (i) to give (ii). If y is even then the second to last term on the right-hand side is $\mathscr{S}(x+y-1,1^2)$ which by (i) is equal to $\mathscr{S}(x+y+1)+\mathscr{S}(x+y-1,2)$. Making this substitution we can again subtract $\mathscr{S}(x+1,1^y)$ and $\mathscr{S}(x,1^{y+1})$ from both sides and this gives the formula for $\mathscr{S}(x,2,1^{y-1})$. ∎

We can now establish the result for the first column of the decomposition matrix of $\mathscr{H}$. Succinctly, the entries in the first column of the decomposition matrix are 1 opposite the hook partitions which have the same 2-core as (n) and 0 elsewhere.

(3.2) THEOREM. (i) *If n is even then* $\mathscr{P}(n) = \sum_{x=1}^{n} \mathscr{S}(x,1^{n-x})$.

(ii) *If n is odd then* $\mathscr{P}(n) = \sum_{\substack{x=1\\ 2\nmid x}}^{n} \mathscr{S}(x,1^{n-x})$.

Proof. We use induction on n. Since the result is trivial when $n=1$ we may assume that $n>1$ and that the result is known for $\mathscr{P}(n-1)$.

First suppose that n is even. Then, by induction,

$$\mathscr{P}(n-1)\uparrow 1 = \sum_{\substack{x=1\\ 2\nmid x}}^{n} \mathscr{S}(x,1^{n-x})\uparrow 1 = \sum_{x=1}^{n}\mathscr{S}(x,1^{n-x}).$$

Since the only 2-regular partitions on the right-hand side are (n) and $(n-1,1)$ we need only show that $\mathscr{D}(n)\subseteq\mathscr{S}(n-1,1)$. However, since $\mathscr{S}(n-1,1)=\mathscr{S}(2,1^{n-2})$ this is a consequence of Lemma 3.1(i), proving (i).

Now suppose that n is odd. Again by induction

$$\begin{aligned}\mathscr{P}(n-1)\uparrow 0 &= \sum_{x=1}^{n} \mathscr{S}(x, 1^{n-x})\uparrow 0 \\ &= \sum_{\substack{x=1\\ 2\nmid x}}^{n} \mathscr{S}(x, 1^{n-x}) + \sum_{x=2}^{n-2} \mathscr{S}(x, 2, 1^{n-x-2}).\end{aligned}$$

There are three 2-regular partitions on the right-hand side; namely, (n), $(n-2, 2)$, and $(n-3, 2, 1)$. By Theorem 1.4, $\mathscr{S}(n-2, 2)$ is irreducible so $\mathscr{D}(n) \not\subseteq \mathscr{S}(n-2, 2)$; consequently there are at least two indecomposables in the sum above. By Lemma 3.1(i), $\mathscr{D}(n) \subseteq \mathscr{S}(x, 1^{n-x})$ whenever x is odd. Also, by Lemma 3.1(ii), $\mathscr{D}(n-2, 2) \subseteq \mathscr{S}(x, 2, 1^{n-x-2})$ when x is odd and $\mathscr{S}(n-3, 2, 1) \subseteq \mathscr{S}(x, 2, 1^{n-x-2})$ when x is even. However, since $\mathscr{S}(n-3, 2, 1) = \mathscr{S}(3, 2, 1^{n-5})$ this shows that $\mathscr{D}(n-2, 2) \subseteq \mathscr{S}(x, 2, 1^{n-x-2})$ for all $2 \le x \le n-2$ and so completes the proof. ∎

We vastly generalize the next result in the Scattering Theorem (6.4) below; however, we need it to get things started.

(3.3) Corollary. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a 2-regular partition such that $\lambda_i \equiv \lambda_{i+1} \pmod 2$ for some $1 \le i \le r-1$. Let μ be the partition with $\mu_j = \lambda_j$ for $j \notin \{i, i+1\}$, $\mu_i = \lambda_i - 1$, and $\mu_{i+1} = \lambda_{i+1} + 1$. Then $d_{\mu\lambda} = 1$.*

Proof. Removing the first $(i-1)$ rows and the first λ_{i+1} columns using Theorem 1.2 shows that $d_{\mu\lambda} = d_{(n-1,1)(n)} = 1$ where $n = \lambda_i - \lambda_{i+1}$. ∎

(3.4) Theorem. (i) *If $n > 1$ is odd then* $\mathscr{P}(n-1, 1) = \sum_{\substack{x=2\\ 2\mid x}}^{n-1} \mathscr{S}(x, 1^{n-x})$.

(ii) *If $n > 2$ is even then*

$$\mathscr{P}(n-1, 1) = \sum_{x=2}^{n-1} \mathscr{S}(x, 1^{n-x}) + \sum_{\substack{x=2\\ 2\mid x}}^{n-2} \mathscr{S}(x, 2, 1^{n-x-2}).$$

Proof. Suppose first that n is odd. By Theorem 3.2

$$\mathscr{P}(n-1)\uparrow 1 = 2\sum_{\substack{x=2\\ 2\mid x}}^{n-1} \mathscr{S}(x, 1^{n-x})$$

so (i) follows.

Now suppose that $n > 2$ is even. Then by (i) we see that

$$\mathscr{P}(n-2, 1)\uparrow 0 = \sum_{x=2}^{n-1} \mathscr{S}(x, 1^{n-x}) + \sum_{\substack{x=2\\ 2\mid x}}^{n-2} \mathscr{S}(x, 2, 1^{n-x-2}).$$

Hence this module is projective and it suffices to show that $d_{(n-2,2)(n-1,1)} \neq 0$ since $(n-1,1)$ and $(n-2,2)$ are the only 2-regular partitions on the right-hand side. However, $d_{(n-2,2)(n-1,1)} = 1$ by Corollary 3.3 so we are done. ∎

Continuing in a similar vein it is not difficult to give explicit formulae for the next few columns of the decomposition matrix (indeed, the proof of Theorem 3.2 constructs the indecomposable $\mathscr{P}(n-2,2)$ for n odd); however, we postpone such considerations until Section 7 where we give explicit formulae for all of the indecomposable modules which correspond to 2-regular partitions of length 2. Before we can do this we need to use the information garnered in the last two sections to build a more general theory.

4. THE GIRTH OF A PROJECTIVE MODULE

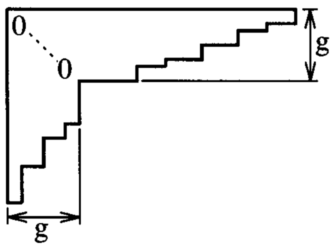
Recall from Definition 1.5 that a partition λ of length r is alternating if $\lambda_i \not\equiv \lambda_{i+1} \pmod 2$ whenever $1 \le i \le r-1$. By Theorem 1.4, if λ is alternating then $\mathscr{S}(\lambda) = \mathscr{D}(\lambda)$. Consequently:

(4.1). Suppose that λ is alternating and that P is a projective $\mathscr{H}$-module containing $\mathscr{S}(\lambda)$. Then $P = \mathscr{P}(\lambda) \oplus P'$ for some $\mathscr{H}$-module P' (possibly 0).

(4.2) Definition. The *girth* of a projective $\mathscr{H}$-module P is the integer g such that

$$g = \max\{k : \text{there exists } \mathscr{S}(\mu) \subseteq P \text{ such that } \mu_k \ge k\}.$$

If P is a projective module of girth g and $\mathscr{S}(\mu) \subseteq P$, then the shape of μ is tightly constrained. The definition is perhaps best illustrated by the diagram below.



The figure is reminiscent of the results in Section 2 on enormous 2-cores; and indeed the importance of girth also derives from the Littlewood–Richardson rule.

(4.3). Note that if P has girth g then $P\uparrow 1$ has girth at most g; whereas $P\uparrow 0$ has girth at most $g+1$. Similarly, $P\downarrow 1$ and $P\downarrow 0$ have girth at most g.

As we shall see the concept of girth is an important one in describing the decomposition matrix of $\mathscr{H}$. Using it we will be able to apply our results on partitions with enormous 2-cores to "smaller" partitions and also extend these results to other partitions where the Littlewood–Richardson rule does not apply directly. We begin by considering staircase partitions.

(4.4) DEFINITION. A partition λ is a *staircase partition* if λ is an alternating partition of the form $(\lambda_1,\dots,\lambda_r,s,\dots,1)$ where the ith part of λ is $r+s-i+1$ for $i=r+1,\dots,r+s$ and $s\geq r-1$.

Note that staircase partitions are 2-quotient separated (see Definition 2.1).

(4.5) THEOREM. *Let $\lambda=(\lambda_1,\dots,\lambda_r,r,\dots,1)$ be a staircase partition. Then $\mathscr{P}(\lambda)$ has girth r.*

Proof. We use induction on r. If $r=1$ then $\lambda=(n-1,1)$ where n is odd so the result is immediate from Theorem 3.4(i). Suppose then that $r>1$.

First note that $\mathscr{P}(\lambda)$ has girth at least r since $\mathscr{S}(\lambda)\subseteq\mathscr{P}(\lambda)$. By induction, $\mathscr{P}(\lambda_2-1,\dots,\lambda_r-1,r-1,\dots,1)$ has girth $r-1$. Since the projective $\mathscr{H}$-module $\mathscr{P}(\lambda_1-1)$ contains only hook partitions by Theorem 3.2, it follows from the Littlewood–Richardson rule that the module

$$P=\mathscr{P}(\lambda_2-1,\dots,\lambda_r-1,r-1,\dots,1)\circ\mathscr{P}(\lambda_1-1)$$

has girth r; consequently, $P\uparrow 1^{2r}$ also has girth at most r by (4.3). Since $\mathscr{S}(\lambda_1-1,\dots,\lambda_r-1,r-1,\dots,1)\subseteq P$ we see that $\mathscr{S}(\lambda)\subseteq P\uparrow 1^{2r}$. The result now follows by (4.1) and (4.3). ∎

More generally, we find:

(4.6) COROLLARY. *Let $\lambda=(\lambda_1,\dots,\lambda_r,s,\dots,1)$ be a staircase partition with $s\geq r-1$. Then $\mathscr{P}(\lambda)$ has girth $\lfloor(s+r+1)/2\rfloor=\lfloor(l(\lambda)+1)/2\rfloor$.*

Proof. The case where $s=r-1$ is proved in the course of theorem. If $s\geq r$ then the girth of $\mathscr{P}(\lambda)$ is at least $\lfloor(s+r+1)/2\rfloor$ since $\mathscr{S}(\lambda)\subseteq\mathscr{P}(\lambda)$.

The result now follows by induction on s using (4.1) and (4.3) since

$$\mathscr{S}(\lambda_1+1,\dots,\lambda_r+1,s+1,s,\dots,1)\subseteq\mathscr{P}(\lambda_1,\dots,\lambda_r,s,\dots,1)\uparrow\alpha^{r+s+1}$$

where $\alpha=r+s \pmod 2$. ∎

The theorem also allows us to obtain a more general, although less precise, result for arbitrary 2-regular partitions.

(4.7) THEOREM. *Let $\lambda=(\lambda_1,\dots,\lambda_r)$ be a 2-regular partition. Then the girth of $\mathscr{P}(\lambda)$ is at most r.*

Proof. If λ is an alternating partition then by (4.1)

$$\mathscr{P}(\lambda)\subseteq\mathscr{P}(\lambda_1+r-1,\dots,\lambda_r+r-1,r-1,\dots,1)\downarrow 0^{2r-1}1^{2r-2}\cdots\alpha^{r+1},$$

where $\alpha=r \pmod 2$. Consequently $\mathscr{P}(\lambda)$ has girth at most r by (4.3) and Corollary 4.6. If λ is not alternating then the result follows by combining (4.3) with the following lemma.

(4.8) LEMMA. *Let λ be any 2-regular partition of length r. Then there exists an alternating partition λ^* of length r such that $\lambda_1^*=\lambda_1$ and $\mathscr{P}(\lambda)\subseteq\mathscr{P}(\lambda^*)\downarrow\alpha_1^{i_1}\cdots\alpha_{r-1}^{i_{r-1}}$ where $\alpha_j=0$ or 1 and $i_j=0$ or $r-j$ $(1\le j\le r-1)$.*

Proof. We use induction on r. When $r=1$ there is nothing to prove so suppose that $r>1$. By induction there exists a partition $\nu=(\nu_2,\dots,\nu_r)$ with $\nu_2=\lambda_2$ and $\mathscr{P}(\lambda_2,\dots,\lambda_r)\subseteq\mathscr{P}(\nu)\downarrow\alpha_2^{i_2}\cdots\alpha_{r-1}^{i_{r-1}}$ where α_j and i_j are as above for $2\le j\le r-1$. Define λ^* to be the partition with $\lambda_1^*=\lambda_1$ and $\lambda_i^*=\nu_i$ if $\lambda_1\not\equiv\lambda_2 \pmod 2$ and $\lambda_i^*=\nu_i+1$ otherwise for $2\le i\le r$. Then λ^* is alternating and it is easy to see that

$$\mathscr{P}(\lambda)\subseteq\mathscr{P}(\lambda_1,\nu_2,\dots,\nu_r)\downarrow\alpha_2^{i_2}\cdots\alpha_{r-1}^{i_{r-1}}\subseteq\mathscr{P}(\lambda^*)\downarrow\alpha_1^{i_1}\cdots\alpha_{r-1}^{i_{r-1}},$$

where $\alpha_1=\lambda_1-1 \pmod 2$ and $i_1=r-1$ if $\lambda_1\equiv\lambda_2 \pmod 2$ and $i_1=0$ otherwise. ∎

(4.9) COROLLARY. *Suppose that $\lambda=(\lambda_1,\dots,\lambda_r)$ is a partition of length r and that $d_{\lambda\mu}>0$ for some 2-regular partition μ. Then*

(i) *if $\lambda_r\ge r$ then $l(\mu)=r$;*

(ii) *if $l(\mu)=l(\lambda)$ then $0\le\lambda_r-\mu_r\le r-1$.*

Proof. If $\lambda_r\ge r$ and $d_{\lambda\mu}\ne 0$ then $\mathscr{P}(\mu)$ has girth at least r so by Theorem 4.7, $l(\mu)\ge r$. On the other hand, $\lambda\trianglelefteq\mu$ so $l(\mu)\le l(\lambda)$ and (i) follows.

Suppose now that $l(\mu)=r$. Then $\mu_r\le\lambda_r$ since $\lambda\trianglelefteq\mu$ so we can use Theorem 1.2 to remove the first μ_r columns from λ and μ which shows that

$$0\ne d_{\lambda\mu}=d_{(\lambda_1-\mu_r,\dots,\lambda_r-\mu_r)(\mu_1-\mu_r,\dots,\mu_{r-1}-\mu_r)}.$$

By Theorem 4.7 $\mathscr{P}(\mu_1 - \mu_r, \dots, \mu_{r-1} - \mu_r)$ has girth at most $r - 1$, so it follows that $\lambda_r - \mu_r \leq r - 1$, proving (ii). ∎

It is not hard to write down an explicit description of the partition λ^* constructed in the proof of Lemma 4.8. In general it appears that the most efficient way to construct the indecomposable $\mathscr{P}(\lambda)$ may be to start with the indecomposable for λ^* and then restrict accordingly. This gives a projective module which contains $\mathscr{P}(\lambda)$ and now one is left with the task of deciding whether there are any other indecomposables contained in this module.

Unfortunately we are not at present able to describe the indecomposables for all alternating partitions. By the proof of Lemma 4.8 the indecomposable module corresponding to any 2-regular partition is contained in some restriction of some $\mathscr{P}(\lambda)$ where λ is a staircase partition; we describe these indecomposables next.

5. STAIRCASE PARTITIONS

With the last section for motivation we now investigate the indecomposable $\mathscr{H}$-modules corresponding to staircase partitions (see Definition 4.4).

(5.1) LEMMA. *Let $\lambda = (\lambda_1, \dots, \lambda_r, r-1, \dots, 1)$ be a staircase partition. Then the only 2-regular partition contained in $\mathscr{P}(\lambda)$ is λ itself. Moreover, if $d_{\mu\lambda} \neq 0$ then $\mu_i \equiv \lambda_i \pmod 2$ and $\mu'_i \equiv \lambda'_i \pmod 2$ for all i where $1 \leq i \leq r$.*

Proof. First note that the 2-core of λ is the partition $(2r-1, 2r-2, \dots, 1)$. If μ is a partition of $|\lambda|$ such that $d_{\mu\lambda} \neq 0$ then the diagram for μ can be obtained from λ by rearranging the dominoes of λ which are outside the 2-core. Since λ is alternating and of the given shape the only movable dominoes appear in the first r rows and are horizontal. Moreover, since $\mathscr{P}(\lambda)$ has girth r by Corollary 4.6, μ must be obtained by a combination of shuffling dominoes between the first r rows of λ and moving horizontal dominoes from the first r rows into the first r columns and placing them there vertically. Consequently, if $d_{\mu\lambda} > 0$ then $\mu_i \equiv \lambda_i \pmod 2$ and $\mu'_i \equiv \lambda'_i \pmod 2$ for all $1 \leq i \leq r$ as claimed.

Now, if dominoes are shuffled only within the first r rows then another alternating partition μ results; so $d_{\mu\lambda} = 0$ by (4.1). Alternatively, since $\mathscr{P}(\lambda)$ has girth r, if any dominoes are moved into the first r columns we obtain a 2-singular partition. Hence λ is the only 2-regular partition in $\mathscr{P}(\lambda)$ and the proof is complete. ∎

In order to proceed further we need to introduce some new notation. As is usual, given an integer k we let $[k] = \{1, 2, \dots, k\}$.

(5.2) *Notation.* (i) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition with $\lambda_r > 0$ and let $I \subseteq [r+1]$ and $J \subseteq [\lambda_1 + 1]$. Then $\lambda^{I \uparrow J}$ is the partition obtained from λ by first adding a node to row i for all $i \in I$ and then adding a node to column j for all $j \in J$ (with the agreement that $\lambda^{I \uparrow J} = 0$ if this is not a partition).

(ii) Similarly, if $I \subseteq [r]$ and $J \subseteq [\lambda_1]$ then $\lambda^{I \downarrow J}$ is the partition where nodes are removed from these rows and columns (where again $\lambda^{I \downarrow J}$ is 0 if this is not a partition).

(iii) If $I = \{i_1, \ldots, i_a\}$ and $J = \{j_1, \ldots, j_b\}$ then we shall sometimes write $\lambda^{i_1 \cdots i_a \uparrow j_1 \cdots j_b}$ instead of $\lambda^{I \uparrow J}$ and similarly for $\lambda^{I \downarrow J}$.

If either superscript I or J is empty then we omit it. We frequently write $\lambda^{(i\downarrow)(j\uparrow)}$ for the partition obtained from λ by moving a node from the end of the ith row to the end of the jth row.

(5.3) Example. Let $\lambda = (17, 14, 13, 8, 3, 2, 1)$. This is a staircase partition of the type we are currently considering. Using our new notation $(18, 15, 14, 9, 4, 3, 2, 1) = \lambda^{[8]\uparrow} = \lambda^{[4]\uparrow[4]} = \lambda^{\uparrow[8]}$ and $\lambda^{2,4\downarrow[3]} = (17, 13^2, 7, 2, 1)$.

(5.4) Theorem. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r, s, \ldots, 1)$ *be a staircase partition with* $s \geq r - 1$. *Then*:

(i) *The only* 2*-regular partition appearing in* $\mathscr{P}(\lambda)$ *is* λ. *Moreover, if* $d_{\mu\lambda} \neq 0$ *then* $\mu_i \equiv \lambda_i \pmod 2$ *for all* $1 \leq i \leq r$ *and* $\mu'_j \equiv \lambda'_j \pmod 2$ *for all* $1 \leq j \leq s+1$.

(ii) $\mathscr{P}(\lambda)\uparrow \alpha^{r+s+1} = (r+s+1)!\mathscr{P}(\lambda^{[r+s+1]\uparrow})$ *where* $\alpha = r + s \pmod 2$.

(iii) $\mathscr{P}(\lambda)\downarrow \beta^{r+s} = (r+s)!\mathscr{P}(\lambda^{[r+s]\downarrow})$ *when* $s \geq r$ *and* $\beta = r + s + 1 \pmod 2$.

(iv) *There is a bijection*

$$\theta = \theta_\lambda : \left\{ \mu \vdash |\lambda| : d_{\mu\lambda} \neq 0 \right\} \to \left\{ \mu \vdash |\lambda^{[r+s+1]\uparrow}| : d_{\mu\lambda^{[r+s+1]\uparrow}} \neq 0 \right\}$$

such that $\theta(\lambda) = \lambda^{[r+s+1]\uparrow}$ *and* $d_{\mu\lambda} = d_{\theta(\mu)\theta(\lambda)}$ *for all* μ. *Explicitly,* $\theta(\mu) = \mu^{[r]\uparrow[s+1]}$.

Proof. We prove (i), (ii), and (iv) by induction on s. By Lemma 5.1, (i) is true when $s = r - 1$. Suppose now that $s \geq r$ and that we know that (i) holds for λ; we show that this implies (ii) and (iv) for λ and (i) for $\lambda^{[r+s+1]\uparrow} = (\lambda_1 + 1, \ldots, \lambda_r + 1, s+1, s, \ldots, 1) = \lambda^{[r]\uparrow[s+1]}$.

Let $\alpha = r + s \pmod 2) \equiv \lambda_1 + 1 \pmod 2$ as in (ii) above and suppose that $d_{\mu\lambda} \neq 0$. By induction, $\mu_i \equiv \lambda_i \pmod 2$ and $\mu'_j \equiv \lambda'_j \pmod 2$ for all for $0 \le i \le r$ and $0 \le j \le s + 1$. Therefore, $0 \neq \mathscr{S}(\mu^{[r]\uparrow[s+1]}) \subseteq \mathscr{P}(\lambda)\uparrow \alpha^{r+s+1}$ and the only places where a node with α's parity may be added to (the diagram of) μ are the first r rows and the first $s+1$ columns. Consequently,

$$(5.5) \qquad \mathscr{S}(\mu)\uparrow \alpha^{r+s+1} = (r+s+1)!\mathscr{S}(\mu^{[r]\uparrow[s+1]}),$$

so (i) holds for $\lambda^{[r+s+1]\uparrow}$ (since $\mathscr{P}(\lambda^{[r+s+1]\uparrow}) \subseteq \mathscr{P}(\lambda)\uparrow \alpha^{r+s+1}$ by (4.1)). Moreover, by (i) μ is 2-regular if and only if $\mu^{[r]\uparrow[s+1]}$ is 2-regular, so $\lambda^{[r+s+1]\uparrow} = \lambda^{[r]\uparrow[s+1]}$ is the only 2-regular partition in $\mathscr{P}(\lambda)\uparrow \alpha^{r+s+1}$. Hence (ii) is established, and (ii) and (5.5) together give (iv). Therefore, we have proved (i), (ii), and (iv) for all λ.

It remains to consider (iii); however, this now follows from the bijection in (iv) together with (ii). ∎

(5.6) EXAMPLE. Suppose that $n > 1$ is odd. Then $(n-1, 1)$ is a staircase partition so $3!\mathscr{P}(n,2,1) = \mathscr{P}(n-1,1)\uparrow 0^3$ by the previous theorem. Therefore, by Theorem 3.4(i),

$$3!\mathscr{P}(n,2,1) = \sum_{\substack{x=2\\ 2\mid x}}^{n-1} \mathscr{S}(x, 1^{n-x})\uparrow 0^3 = 3!\sum_{\substack{x=2\\ 2\mid x}}^{n-1} \mathscr{S}(x+1, 2, 1^{n-x}).$$

More generally, applying Theorem 5.4 $(k+1)$ times shows that

$$\mathscr{P}(n+k, k+2, k+1, \dots, 1) = \sum_{\substack{x=2\\ 2\mid x}}^{n-1} \mathscr{S}(x+k+1, k+2, \dots, 2, 1^{n-x})$$

for all $k \ge 1$ when $n > 1$ is odd.

The example is a good illustration of our next result because with successive applications of Theorem 5.4 we obtain partitions with larger and larger 2-cores which all have the same 2-weight as $(n-1, 1)$. Consequently, for large enough k all of the partitions appearing in $\mathscr{P}(n+k, k+2, k+1, \dots, 1)$ have enormous 2-cores and so this example becomes a special case of Theorem 2.12.

Let $\lambda = (\lambda_1, \dots, \lambda_r, s, \dots, 1)$ be a staircase partition with $s \ge r - 1$. Then λ is 2-quotient separated (see Definition 2.1), and so we may write $\lambda = (\lambda^h; (0))_{r+s}$ (note that $l(\tilde{\lambda}) = r + s$).

(5.7) THEOREM. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r, s, \ldots, 1)$ *be a staircase partition of* n *with* $s \geq r - 1$ *and write* $\lambda = (\lambda^h; (0))_{r+s}$ *as above. Then for all partitions* μ *of* n

$$d_{\mu\lambda} = \begin{cases} \alpha^{\lambda^h}_{\mu^h \mu^v} & \text{if } \mu = (\mu^h; \mu^v)_{r+s} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $d_{\mu\lambda} \neq 0$. Then by Theorem 5.4(i) μ is 2-quotient separated and we may write $\mu = (\mu^h; \mu^v)_{r+s}$. Let θ be the bijection of Theorem 5.4(iv). Then $\theta(\mu) = \mu^{[r]\uparrow[s+1]}$, so

$$\theta\big((\mu^h; \mu^v)_{r+s}\big) = (\mu^h; \mu^v)_{r+s+1}.$$

Consequently, by applying the map θ sufficiently many times we can "inflate" all of the partitions in $\mathscr{P}(\lambda)$ to partitions with enormous 2-cores. Since the map θ preserves decomposition numbers the result now follows by Theorem 2.12. ∎

An example of a calculation using Theorem 5.7 is given in Lemma 7.3 where it is used to give an explicit formula for the indecomposable $\mathscr{H}$-modules of the form $\mathscr{P}(\lambda_1, \lambda_2, 1)$ where $(\lambda_1, \lambda_2, 1)$ is a staircase partition.

(5.8) COROLLARY. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r, s, \ldots, 1)$ *be a staircase partition with* $s \geq r - 1$ *and let* g *be the girth of* $\mathscr{P}(\lambda)$. *Suppose that* $d_{\mu\lambda} \neq 0$. *Then*

$$\mu_i \leq \lambda_i \text{ and } \mu'_i \geq \lambda'_i \qquad \text{for all } 1 \leq i \leq g.$$

Indeed, $\mu'_i = \lambda'_i = r + s + 1 - i$ *for all* $r < i \leq s + 1$.

Proof. In order to find which $\mathscr{S}(\mu)$ occur in $\mathscr{P}(\lambda)$ all we need to do is move horizontal dominoes from the first r rows of λ (outside the 2-core) into the first r columns in all possible ways allowed by the inverse Littlewood-Richardson rule. In particular, nodes may be added only to the first r columns of λ in forming μ. The result follows. ∎

(5.9) THEOREM. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r, s, \ldots, 1)$ *be a staircase partition of* n, *with* $s \geq r - 1$, *and suppose that* $0 \leq k \leq r + s + 1$. *Then the following hold.*

(i) *Suppose that* σ *is a partition of* $n+k$. *Then* $d_{\sigma\lambda^{[k]\downarrow}} = d_{\mu\lambda}$ *if* $\sigma = \mu^{I\uparrow J}$ *for some partition* μ *of* n *and some* $I \subseteq [r]$ *and* $J \subseteq [s+1]$ *with* $|I| + |J| = k$, *and* $d_{\sigma\lambda^{[k]\uparrow}} = 0$ *otherwise*.

(ii) *If* $\sigma = \lambda^{I\uparrow}$ *for some* $I \subseteq [r+s+1]$, *and* $|I| = k$, *then* $d_{\sigma\lambda^{[k]\uparrow}} = 1$; *moreover, every* 2*-regular partition* σ *such that* $d_{\sigma\lambda^{[k]\uparrow}} \neq 0$ *has this form*.

(iii) *Let* $\alpha = r + s \pmod 2$. *Then* $\mathscr{P}(\lambda)\uparrow \alpha^k = k!\mathscr{P}(\lambda^{[k]\uparrow})$.

(iv) *Let* $\beta = r + s + 1 \pmod 2$. *If* $s \geq r$ *and* $0 \leq k \leq r + s$ *then* $\mathscr{P}(\lambda)\downarrow \beta^k = k!\mathscr{P}(\lambda^{\downarrow[k]})$.

Proof. Let σ be a partition of $n+k$ and $\alpha = r+s \pmod 2$. By the congruences of Theorem 5.4(i), $\mathscr{S}(\sigma) \subseteq \mathscr{P}(\lambda)\uparrow \alpha^k$ if and only if $\sigma = \mu^{I\uparrow J}$ where $d_{\mu\lambda} \neq 0$, $I \subseteq [r]$, $J \subseteq [s+1]$, and $|I| + |J| = k$; moreover, $\mathscr{S}(\sigma)$ appears with multiplicity $k!d_{\mu\lambda}$ in $\mathscr{P}(\lambda)\uparrow \alpha^k$.

Assume that $d_{\mu\lambda} \neq 0$ for some partition $\mu \neq \lambda$. We first show that all partitions of the form $\mu^{I\uparrow J}$ are 2-singular. By Corollary 4.6, $g = \lfloor (r+s+1)/2 \rfloor$ is the girth of $\mathscr{P}(\lambda)$. Since $\mu \neq \lambda$, $\mu'_j \geq \lambda'_j + 2$ for some j with $1 \leq j \leq g$ by Corollary 5.8. If $g \leq s$ (i.e., $s > r-1$), then $\mu'_{g+1} = \lambda'_{g+1}$ and hence $\mu'_i \geq \mu'_{i+1} + 3$ for some i with $1 \leq i \leq g$, and so $\mu^{I\uparrow J}$ is 2-singular. On the other hand, if $g > s$ then $s = r-1$ and if $\mu^{I\uparrow J}$ is 2-regular then we must have $\mu'_i = 2r - i + 2$ for all $i \in [r]$. However, such μ become 2-regular only after a node with 2-residue 0 is added to the end of the $(r+1)$st row of μ; as $\alpha = 1$ when $s = r-1$ it follows that $\mathscr{S}(\mu)\uparrow \alpha^k$ contains only 2-singular partitions.

We next show that if $\sigma = \lambda^{I\uparrow}$, where $I \subseteq [r+s+1]$ and $|I| = k$, then $d_{\sigma\lambda^{[k]\uparrow}} = 1$. By way of contradiction, let σ be a partition, first in the lexicographic order, for which this is false. Then $\sigma \neq \lambda^{[k]\uparrow}$ and σ must be 2-regular with $k!\mathscr{P}(\sigma) \subseteq \mathscr{P}(\lambda)\uparrow \alpha^k$. Now if, when we construct σ from λ, we were to add nodes only to the first s columns of λ then σ would be 2-singular. Therefore, at least one node has been added to one of the first $r+1$ rows of λ. Hence there exists a $j \in [r]$ such that $\sigma_j = \lambda_j$ and $\sigma_{j+1} = \lambda_{j+1} + 1$. Also, $\sigma_j > \sigma_{j+1} + 1$ since σ is 2-regular and λ is alternating. Let $\rho = \sigma^{(j\downarrow)(j+1\uparrow)}$; then $\rho \neq 0$ and $d_{\rho\sigma} = 1$ by Corollary 3.3. Therefore $\mathscr{S}(\rho) \subseteq \mathscr{P}(\sigma) \subseteq \mathscr{P}(\lambda)\uparrow \alpha^k$. In particular, $\rho_i \leq \lambda_i + 1$ for $1 \leq i \leq r$ by Corollary 5.8, and $\rho_{r+1} \leq r+1$ since $\mathscr{P}(\lambda)\uparrow \alpha^k$ has girth at most $r+1$. But $\rho_{j+1} = \lambda_{j+1} + 2$, which is a contradiction. Therefore, $d_{\sigma\lambda^{[k]\uparrow}} = 1$, as required.

All parts of the theorem now follow. ∎

(5.10) *Remark.* Combining the results of Theorem 5.4, Theorem 5.7, and Theorem 5.9 we see that if $\lambda = (\lambda_1, \ldots, \lambda_r, r-1, \ldots, 1)$ is any staircase partition then the decomposition numbers $d_{\mu\lambda}$ are determined by the (inverse) Littlewood–Richardson rule and these decomposition numbers determine the decomposition numbers for the indecomposable $\mathscr{H}$-modules corresponding to the partitions

$$\lambda, \lambda^{[1]\uparrow}, \lambda^{[2]\uparrow}, \ldots, \lambda^{[r+s+1]\uparrow}, \lambda^{([r+s+1]\uparrow)([1]\uparrow)}, \ldots, \lambda^{([r+s+1]\uparrow)([r+s+2]\uparrow)}, \ldots .$$

6. THE SCATTERING THEOREM

We now apply the results from the previous section to 2-regular partitions of arbitrary shape.

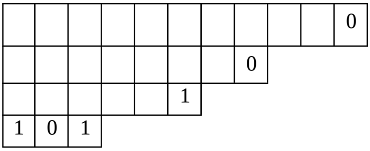
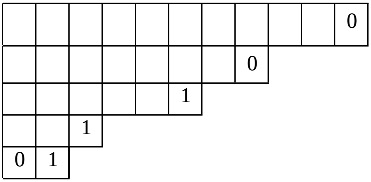
(6.1) THEOREM. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_r > 0$, is a 2-regular partition of n for which there exists an integer b with $0 \le b \le r-1$ such that both $(\lambda_1, \ldots, \lambda_b)$ and $(\lambda_{b+1}, \ldots, \lambda_r)$ are alternating partitions and $\lambda_b - \lambda_{b+1}$ is even (if $b > 0$). Let μ be any partition of n.*

(i) *Suppose that λ_r is odd and that $l(\mu) \le r+1$. Then $d_{\mu\lambda} \ne 0$ if and only if $\mu = \lambda^{(I\downarrow)(J\uparrow)}$ for two sets $I \subseteq [b]$ and $J \subseteq \{b+1, \ldots, r+1\}$ such that $|I| = |J|$. If $d_{\mu\lambda} \ne 0$ then $d_{\mu\lambda} = 1$.*

(ii) *Suppose that λ_r is even and $l(\mu) = r$. Then $d_{\mu\lambda} \ne 0$ if and only if $\mu = \lambda^{(I\downarrow)(J\uparrow)}$ for two sets $I \subseteq [b]$ and $J \subseteq \{b+1, \ldots, r\}$ such that $|I| = |J|$. If $d_{\mu\lambda} \ne 0$ then $d_{\mu\lambda} = 1$.*

Before we prove theorem we give an example to illustrate it.

(6.2) EXAMPLE. The result concerns partitions which can be "cut" at the bth row to give two alternating partitions. It has a "pictorial" description using 2-residues and diagrams. Consider the two partitions $(11, 8, 6, 3)$ and $(11, 8, 6, 3, 2)$ and their diagrams:

For both partitions $b = 2$. The theorem says, for example, that $d_{(10,7,7,3,1)(11,8,6,3)} = 1$ and $d_{(10,8,6,4,2)(11,8,6,3,2)} = 1$. To prove the theorem for the first partition we add three columns of lengths 5, 6, and 7 to the left of the diagram of $(11, 8, 6, 3)$ so that we can consider the partition $(14, 11, 9, 6, 3, 2, 1)$ and so apply Theorem 5.9. For the second partition, since we are only interested in partitions μ of length 5, we can remove the first column from λ and then we are back in case (i).

Proof of Theorem 6.1. Suppose first that λ_r is odd and $l(\mu) \leq r+1$. Then by Theorem 1.2, $d_{\mu\lambda} = d_{\mu^*\lambda^*}$ where

$$\mu^* = (\mu_1 + r - 1, \ldots, \mu_r + r - 1, \mu_{r+1} + r - 1, r - 2, \ldots, 1)$$

and

$$\lambda^* = (\lambda_1 + r - 1, \ldots, \lambda_r + r - 1, r - 1, \ldots, 1)$$

(so we have added a "wedge" of $(r-1)$ columns to the left of λ and μ to make them resemble staircase partitions). Then $\lambda^* = \nu^{[b]\uparrow}$ where $\nu = (\nu_1, \ldots, \nu_r, r-1, \ldots, 1)$ is a staircase partition. Therefore, by Theorem 5.9(ii), $d_{\mu\lambda} \neq 0$ if and only if $\mu^* = \nu^{L\uparrow}$ for some $L \subseteq [r+1]$ with $|L| = b$, in which case $d_{\mu\lambda} = 1$. Let $I = [b] \setminus L$ and $J = L \setminus [b]$. Then $\mu^* = \lambda^{*(I\downarrow)(J\uparrow)}$ and $|I| = |J|$. Moreover, since λ^* and μ^* agree on the first $(r-1)$ columns, $\mu = \lambda^{(I\downarrow)(J\uparrow)}$ as required.

Now suppose that λ_r is even and $l(\mu) = r$. Removing the first column from λ and μ using Theorem 1.2 shows that $d_{\mu\lambda} = d_{(\mu_1 - 1, \ldots, \mu_r - 1)(\lambda_1 - 1, \ldots, \lambda_r - 1)}$, so part (ii) follows from part (i). ∎

The following result on alternating partitions will be needed later:

(6.3) COROLLARY. *Let λ be an alternating partition of length r and let μ be any partition with $l(\mu) \leq r+1$.*

(i) *Suppose that λ_r is odd. Then $d_{\mu\lambda} \neq 0$ if and only if $\mu = \lambda$.*

(ii) *Suppose that λ_r is even. Then $d_{\mu\lambda} \neq 0$ if and only if $\mu = \lambda$ or $\mu = \lambda^{(i\downarrow)(r+1\uparrow)}$ for some $i \in [r]$, in which case $d_{\mu\lambda} = 1$.*

Proof. The case where λ_r is odd is a direct application of Theorem 6.1, taking the first partition $(\lambda_1, \ldots, \lambda_b)$ to be empty (i.e., $b = 0$).

Now suppose that λ_r is even. This amounts to saying that a node can be moved from the end of any row of λ down to the $(r+1)$st row without changing its 2-residue. To prove part (ii), add a new first column of length $r+1$ to λ, apply part (i) of Theorem 6.1, and then remove the first column. ∎

Using Theorem 1.2 we can give a small generalization of Theorem 6.1 which is useful for practical applications. In this guise the result is a variation on a result of Carter and Payne [1] for the symmetric group.

In the statement of the theorem, when $c = r+1$ the condition that $(\lambda_{b+1}, \ldots, \lambda_c)$ is alternating is to be interpreted as meaning that $\lambda_i - \lambda_{i+1}$ is odd for $b+1 \leq i \leq r$.

(6.4) THEOREM (Scattering Theorem). *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_r > 0$, is a* 2*-regular partition for which there exist integers* $1 \leq a \leq$

$b < c \le r+1$ such that both $(\lambda_a, \ldots, \lambda_b)$ and $(\lambda_{b+1}, \ldots, \lambda_c)$ are alternating partitions and $\lambda_b \equiv \lambda_{b+1} \pmod 2$. Let $I \subseteq \{a, \ldots, b\}$ and $J \subseteq \{b+1, \ldots, c\}$ be two sets such that $|I| = |J|$ and set $\mu = \lambda^{(I\downarrow)(J\uparrow)}$. Then $d_{\mu\lambda} = 1$.

Proof. Using Theorem 1.2 to remove extra rows and columns we can reduce to the case where $a = 1$ and $c = r+1$ which is the situation considered in Theorem 6.1(i). ∎

Although the statement looks somewhat formidable all the result says is that Theorem 6.1 can be applied to adjacent alternating partitions "inside" a given partition. For example, if $\lambda = (15, 12, 11, 8, 6, 3, 2, 1)$ then λ "contains" the alternating partitions $(15, 12)$, $(11, 8)$, and $(6, 3, 2, 1)$. Applying the Scattering Theorem to the second and third of these partitions, for example, shows that $d_{(15, 12, 10, 8, 6, 4, 2, 1)\lambda} = 1$.

The Scattering Theorem is quite powerful both as a theoretical and as a practical tool. We first use it to characterize the irreducible Specht modules. The idea of the proof is simple; for example, if $\lambda = (11, 8, 6, 3, 2)$ is the second partition considered in Example 6.2 then we "pick up" the three ones at the end of the last three rows of λ and move them to the end of the first three rows of λ. The Scattering Theorem shows that $d_{(11,8,6,3,2)(12,9,6,2,1)} = 1$, so $\mathscr{S}(\lambda)$ is not irreducible.

(6.5) COROLLARY. *Let λ be any 2-regular partition. Then $\mathscr{S}(\lambda)$ is irreducible if and only if λ is alternating.*

Proof. If λ is alternating then by Theorem 1.4 $\mathscr{S}(\lambda)$ is irreducible; so suppose that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is not alternating. Then there exists an i such that $\lambda_{i-1} \equiv \lambda_i \pmod 2$ where $1 < i \le r$. Choose i to be minimal with this property (so i is the first row to end with a 2-residue different from that at the end the first row of λ), and let j be the smallest number such that $\lambda_i - i \equiv \lambda_j - j \pmod 2$ and $\lambda_{j+1} \equiv \lambda_j \pmod 2$ (so the 2-residue at the end of the ith and jth rows are equal but the 2-residue at the end of the $(j+1)$th row is different). If there is no such j take $j = l(\lambda)$. Now let $I = \{i, \ldots, j\}$ and $J = [j - i + 1]$ and let $\mu = \lambda^{(I\downarrow)(J\uparrow)}$. Then μ is a 2-regular partition with $\mu \ne \lambda$ and $d_{\lambda\mu} = 1$ by the Scattering Theorem. That is to say, $\mathscr{D}(\mu) \subseteq \mathscr{S}(\lambda)$; so $\mathscr{S}(\lambda)$ is not irreducible. ∎

By Theorem 5.7 we know the indecomposable module $\mathscr{P}(\lambda)$ for all staircase partitions $\lambda = (\lambda_1, \ldots, \lambda_r, s, \ldots, 1)$, where $s \ge r - 1$, and in Theorem 5.9 we described how to restrict (and induce) these modules whenever $s \ge r$. Using the Scattering Theorem we next investigate the restriction of $\mathscr{P}(\lambda)$ for partitions with $s = r - 1$. It turns out that we can go down one more "level" without losing indecomposability; however, restricting past this limit will, in general, yield more than one indecomposable.

(6.6) THEOREM. *Let $\lambda = (\lambda_1, \dots, \lambda_r, r-1, \dots, 1)$ be a staircase partition. Then $\mathscr{P}(\lambda){\downarrow}0^k = k!\mathscr{P}(\lambda^{\downarrow[k]})$ for all $0 \le k \le 2r-1$.*

Proof. If $\lambda_r = r$ the result is a special case of Theorem 5.9, so assume that $\lambda_r > r$. Let μ be a partition such that $d_{\mu\lambda} > 0$. Then $\mu_i \equiv \lambda_i \pmod 2$ and $\mu_i' \equiv \lambda_i' \pmod 2$ for $1 \le i \le r$ by Lemma 5.1. Therefore, we may write $\mu = (\mu^h; \mu^v)_c$ and

$$\mathscr{S}(\mu){\downarrow}0^k = k!\sum_{I,J}\mathscr{S}(\mu^{I\downarrow J})$$

where the sum is over all subsets $I, J \subseteq [r]$ such that $|I| + |J| = k$. Consequently $\mathscr{S}(\mu){\downarrow}0^k$ contains a 2-regular partition if and only if $\mu = \lambda$ or μ is of the form $(\mu_1, \dots, \mu_r, r, r, r-1, \dots, 1)$ and $k \ge r$. In the latter case $\mu^v = (1^r)$ so the Littlewood–Richardson rule, via Theorem 5.7, forces μ to be the partition $(\lambda_1 - 2, \dots, \lambda_r - 2, r, r, r-1, \dots, 1)$ and $d_{\mu\lambda} = 1$. Now, if ν is a 2-regular partition such that $\mathscr{S}(\nu) \subseteq \mathscr{S}(\mu){\downarrow}0^k$ then there exists $1 \le i \le r$ such that $\nu_i = \mu_i$; choose i to be maximal with this property. By the Scattering Theorem, $d_{\nu^{(i\downarrow)(r+1\uparrow)}\nu} = 1$, so the girth of $\mathscr{P}(\nu)$ is at least $r+1$. However, $\mathscr{P}(\lambda){\downarrow}0^k$ has girth at most r by (4.3) so $\mathscr{P}(\nu) \not\subseteq \mathscr{P}(\lambda){\downarrow}0^k$.

It remains to show that the indecomposable $\mathscr{H}$-modules corresponding to the 2-regular partitions in $\mathscr{S}(\lambda){\downarrow}0^k$ do not split off. However, if $\mu \ne \lambda^{\downarrow[k]}$ and μ is a 2-regular partition such that $\mathscr{S}(\mu) \subseteq \mathscr{S}(\lambda){\downarrow}0^k$ then $\mu = \lambda^{([k]\downarrow)(I\uparrow)}$ for some set I with $|I| = k$. Therefore, it follows by the Scattering Theorem that $d_{\mu\lambda} = 1$. ∎

(6.7) EXAMPLE. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1)$ be a 2-regular partition where λ_1 is odd and λ_2 and λ_3 are even. Then, by the theorem,

$$3!\mathscr{P}(\lambda) = \mathscr{P}(\lambda_1, \lambda_2, \lambda_3 + 1, 2, 1){\downarrow}0^3.$$

Moreover, by the proof of the theorem the following partitions are in $\mathscr{P}(\lambda)$, each appearing with a multiplicity of 1,

$$\begin{array}{ll}
(\lambda_1, \lambda_2, \lambda_3, 1) & (\lambda_1, \lambda_2 - 1, \lambda_3 + 1, 1) \\
(\lambda_1, \lambda_2 - 1, \lambda_3, 2) & (\lambda_1 - 1, \lambda_2, \lambda_3 + 1, 1) \\
(\lambda_1 - 1, \lambda_2, \lambda_3, 2) & (\lambda_1 - 1, \lambda_2 - 1, \lambda_3 + 1, 2) \\
(\lambda_1 - 1, \lambda_2 - 1, \lambda_3, 2, 1) & (\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 1, 3, 2, 1);
\end{array}$$

and the 2-regular partitions in $\mathscr{P}(\lambda)$ are the 2-regular partitions among those listed. Note that the last of these partitions, $(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 1, 3, 2, 1)$, comes from restricting the 2-singular partition $(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 1, 3^2, 2, 1)$ in $\mathscr{P}(\lambda_1, \lambda_2, \lambda_3 + 1, 2, 1)$. The proof of Theorem 6.6 is largely concerned with showing that the indecomposable modules corresponding to such partitions do not split off.

7. APPLICATIONS TO 2-REGULAR PARTITIONS WITH 2 PARTS

Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ we write $\lambda \equiv (\alpha_1, \dots, \alpha_r) \pmod 2$ where $\alpha_i = \lambda_i \pmod 2$. The first thing that one notices from the theorem below is that the indecomposables for 2-part partitions are essentially determined by $\lambda \pmod 2$. As we shall see this is also the case for 2-regular partitions of length 3 (with "small" exceptions).

First note that if (k, l) is any 2-regular partition with at least one odd part then Theorem 6.6 constructs the indecomposable $\mathscr{P}(k, l)$ directly. The remaining case is also easy.

(7.1) THEOREM. *Let* $\lambda = (k, l)$ *be any* 2*-regular partition of length* 2.

(i) *If* $\lambda \equiv (1, 0) \pmod 2$ *then* $\mathscr{P}(\lambda) = \mathscr{P}(k, l, 1)\downarrow 0$ *and the only* 2*-regular partitions in* $\mathscr{P}(\lambda)$ *are the* 2*-regular partitions amongst* (k, l), $(k, l-1, 1)$, *and* $(k-1, l, 1)$.

(ii) *If* $\lambda \equiv (1, 1) \pmod 2$ *then* $2!\mathscr{P}(\lambda) = \mathscr{P}(k, l+1, 1)\downarrow 0^2$ *and the only* 2*-regular partitions in* $\mathscr{P}(\lambda)$ *are the* 2*-regular partitions amongst* (k, l), $(k-1, l+1)$, $(k-1, l, 1)$, *and* $(k-2, l-1, 2, 1)$.

(iii) *If* $\lambda \equiv (0, 1) \pmod 2$ *then* $3!\mathscr{P}(\lambda) = \mathscr{P}(k+1, l+1, 1)\downarrow 0^3$ *and the only* 2*-regular partitions in* $\mathscr{P}(\lambda)$ *are the* 2*-regular partitions amongst* (k, l), $(k-1, l-2, 2, 1)$, *and* $(k-2, l-1, 2, 1)$.

(iv) *If* $\lambda \equiv (0, 0) \pmod 2$ *then* $\mathscr{P}(\lambda) = \mathscr{P}(k-1, l)\uparrow 1$ *and the only* 2*-regular partitions in* $\mathscr{P}(\lambda)$ *are the* 2*-regular partitions amongst* (k, l), $(k, l-1, 1)$, $(k-1, l+1)$, $(k-1, l-1, 2)$, $(k-1, l-2, 2, 1)$, $(k-2, l+1, 1)$, $(k-2, l, 2)$, $(k-3, l, 2, 1)$.

Moreover, in every case $d_{\mu\lambda} = 1$ *for the partitions* μ *listed above.*

Proof. The first three parts of theorem are special cases of Theorem 6.6; so it remains to consider part (iv). Let $P = \mathscr{P}(k-1, l)\uparrow 1$; then it is straightforward to check that (k, l), $(k, l-1, 1)$, $(k-1, l+1)$, $(k-1, l-1, 2)$, $(k-1, l-2, 2, 1)$, $(k-2, l+1, 1)$, $(k-2, l, 2)$, and $(k-3, l, 2, 1)$ is a complete list of the 2-regular partitions in P (each with multiplicity 1). It suffices then to show for each of these partitions $\mu \neq \lambda$ that if μ is a 2-regular partition then the corresponding indecomposable $\mathscr{P}(\mu)$ is not contained in P. This is clear if μ is $(k, l-1, 1)$ or $(k-1, l+1)$ since then $d_{\mu\lambda} = 1$ by the Scattering Theorem (6.4). If $\mu = (k-2, l+1, 1)$ then by the Scattering Theorem $d_{(k-3, l+1, 2)\mu} = 1$; however, $\mathscr{S}(k-3, l+1, 2) \not\subseteq P$ so therefore $\mathscr{P}(\mu) \not\subseteq P$. Finally, if μ is one of the remaining partitions then consider $(\mu_1 - 2, \mu_2 - 2)$. By parts (i)–(iv) and induction $d_{\eta(\mu_1-2, \mu_2-2)} = 1$ for some partition η of the form $(\eta_1, \eta_2, 1)$. If $\mu = (k-1, l-2, 2)$ or $(k-2, l, 2)$ then by Theorem 1.2, $d_{(\eta_1+2, \eta_2+2, 3)\mu} =$

$d_{\eta(\mu_1-2,\mu_2-2)} = 1$ so $\mathscr{P}(\mu)$ has girth at least 3. Similarly, if $\mu = (k-1, l-2, 2, 1)$ or $(k-3, l, 2, 1)$ then $\mathscr{P}(\mu)$ also has girth at least 3. However, P has girth at most 2 by (4.3) and Corollary 4.6; so $\mathscr{P}(\mu) \not\subseteq P$ and the result follows. ∎

Note that the part of the decomposition matrix of $\mathscr{H}$ corresponding to 2-regular partitions of length 2 is

$$\begin{array}{l} (n) \\ (n-1,1) \\ (n-2,2) \\ \vdots \end{array} \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} n \text{ odd} \qquad \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{pmatrix} n \text{ even}$$

(the unlabelled entries are 0 and the rows and columns are indexed in the obvious way by the partitions on the left). This observation allows us to calculate the dimensions of the irreducible $\mathscr{H}$-modules corresponding to 2-regular partitions with 2 parts. We give the dimensions in terms of the dimensions of Specht modules.

(7.2) COROLLARY. *Let $k > l$ and let $n = k + l$. Then*

(i) *if n is odd $\mathscr{S}(k,l) = \mathscr{D}(k,l)$;*

(ii) *if n is even $\mathscr{S}(k,l) = \mathscr{D}(k,l) + \mathscr{D}(k+1, l-1)$.*

Consequently,

$$\dim \mathscr{D}(k,l) = \begin{cases} \dim \mathscr{S}(k,l) & \text{if } n \text{ is odd} \\ \dim \mathscr{S}(k-1,l) & \text{if } n \text{ is even}. \end{cases}$$

Proof. If n is odd then (k,l) is an alternating partition so $\mathscr{S}(k,l)$ is irreducible. If n is even then by Theorem 7.1 $\mathscr{S}(k,l) = \mathscr{D}(k+1, l-1) + \mathscr{D}(k,l)$. Restricting this equation to the Hecke algebra $\mathscr{H}_{n-1}$ shows that

$$\mathscr{S}(k, l-1) + \mathscr{S}(k-1, l) = \mathscr{D}(k+1, l-1)\downarrow + \mathscr{D}(k,l)\downarrow,$$

from which it is evident by induction on l that $\mathscr{D}(k,l)\downarrow = \mathscr{S}(k-1,l) = \mathscr{D}(k-1,l)$. ∎

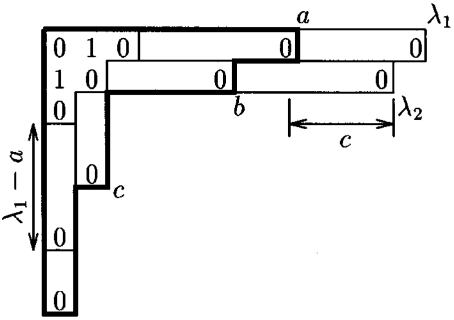
By Theorem 7.1, in order to calculate the decomposition numbers for any projective indecomposable corresponding to a 2-regular partition of length 2 all that is needed is the Branching Theorem and the Littlewood–Richardson rule (the Littlewood–Richardson rule being used to calculate the decomposition numbers for alternating partitions of the form $(k, l, 1)$ by Theorem 5.7). Using this information it is possible to give explicit formulae for the indecomposables corresponding to 2-regular partitions with 2 parts. We begin with the lemma:

(7.3) LEMMA. *Let* $(\lambda_1, \lambda_2, 1)$ *be an alternating partition of* n. *Then*

$$\mathscr{P}(\lambda_1, \lambda_2, 1) = \sum_{\substack{a=3\\ 2\nmid a}}^{\lambda_1} \sum_{\substack{b=2\\ b\le a\\ 2|b}}^{\lambda_2} \sum_{\substack{c=\lambda_2+1-a\\ 0\le c\le \lambda_2-b\\ 2|c}}^{\lambda_1-a} \mathscr{S}(a, b, 2^c, 1^{n-a-b-2c}).$$

Note, for example, that in this sum the partition $(9, 2^7, 1^8)$ would be written as $(9, 2, 2^6, 1^8)$. Throughout the proof the reader will probably find it helpful to refer to (7.4).

Proof. Suppose that $d_{\mu\lambda} > 0$. By Theorem 5.7 we know that the partition μ is of the form $(a, b, 2^c, 1^{n-a-b-2c})$ where $2 \nmid a$, $2|b$, $2|c$, $a \le \lambda_1$, $b \le a$, and $b \le \lambda_2$. By the Littlewood–Richardson rule the $\lambda_1 - a$ nodes moved from the first row of λ must appear in the first column of μ and so $c \le \lambda_1 - a$. On the other hand the nodes in the second column of μ must have come from the second row of λ so $c \le \lambda_2 - b$. Also $c \ge \lambda_2 + 1 - a$ since at most $\lambda_1 - b - 1$ nodes can appear in the first column of μ (outside the 2-core). Finally, as a, b, and c vary subject to these bounds it is easy to see using the Littlewood–Richardson rule that for all possible choices the corresponding partitions μ appear with multiplicity 1. ∎

$\mu = (a, b, 2^c, 1^{n-a-b-2c})$ 

(7.4)

We now give an explicit description of the indecomposable modules of $\mathscr{H}$ which correspond to 2-regular partitions of length 2. We state the theorem as three separate results since the combined statement is quite lengthy. In all cases the results can be proved in a purely mechanical fashion using Theorem 7.1 and Lemma 7.3; by referring to (7.4) we can to some extent avoid this brute force approach.

(7.5) THEOREM. *Suppose that $\lambda = (k,l)$ is an alternating partition of n. Then*

$$\mathscr{P}(k,l) = \sum_{\substack{x=l+1\\ x\equiv k \pmod 2}}^{k} \mathscr{S}(x,1^{n-x}) + \sum_{x=2}^{k}\sum_{\substack{y=2\\ y\le x}}^{l}\sum_{\substack{z=1-x\\ 0\le z\le l-y}}^{k-x} \mathscr{S}(x,y,2^z,1^{n-x-y-2z}),$$

where in the multiple sum

(i) *if $x \equiv l \pmod 2$ then $y \equiv z \equiv l \pmod 2$*

(ii) *if $y \equiv k \pmod 2$ then $x \equiv k \pmod 2$ and $z \equiv l \pmod 2$.*

Proof. Let $\lambda = (k,l)$ be an alternating partition. Suppose first that $\lambda \equiv (1,0) \pmod 2$. By Theorem 7.1(i) $\mathscr{P}(k,l) = \mathscr{P}(k,l,1)\downarrow 0$ and $\mathscr{P}(k,l,1)$ is described explicitly in Lemma 7.3. Let $S = \mathscr{S}(a,b,2^c,1^{n+1-a-b-2c})$ be a general term in $\mathscr{P}(k,l,1)$; then $S\downarrow 0 \subseteq \mathscr{P}(k,l)$ and by (7.4) it is clear that

$$S\downarrow 0 = \mathscr{S}(a,b,2^c,1^{n-a-b-2c}) + \mathscr{S}(a,b,2^{c-1},1^{n+2-a-b-2c}) + \mathscr{S}(a,b-1,2^c,1^{n+1-a-b-2c}) + \mathscr{S}(a-1,b,2^c,1^{n+1-a-b-2c}).$$

Consider these terms written in the form $\mathscr{S}(x,y,2^z,1^{n-x-y-2z})$. Now a is odd and b and c are even. Therefore, since only one node was removed from S it follows that if x is even (i.e., $x \equiv l \pmod 2$), then both y and z are also even in accordance with (i). Similarly, if y is odd then x is odd and z is even giving (ii). Also note that if x is odd and y is even then there is no parity restriction on z; so (i) and (ii) are the only parity restrictions on x, y, and z (here and below we leave the hook partitions in $\mathscr{P}(\lambda)$ as an exercise for the reader). It is now a simple exercise to verify that the bounds on x, y, and z are as stated. Since it is clear that all of these Specht modules appear with multiplicity 1 the result follows when $\lambda \equiv (1,0) \pmod 2$.

When $\lambda \equiv (0,1)$, $3!\mathscr{P}(\lambda) = \mathscr{P}(k+1,l+1,1)\downarrow 0^3$ and a similar argument shows that (i) and (ii) are the only restrictions on the parity of x, y, z. It is straightforward to check that the upper and lower bounds upon x, y, and z are as given. For example, $y \le l$ because $b \le l+1$ in (7.4) and if $b = l+1$ then $c = 0$ so a node must be removed from the second row of $\mathscr{S}(a,b,2^c,1^{n+3-a-b-2c})$ when $\mathscr{P}(k+1,l+1,1)$ is 0-restricted three times. ∎

(7.6) THEOREM. *Suppose that* $\lambda = (k, l) \equiv (1,1) \pmod 2$ *is a* 2*-regular partition of* n. *Then*

$$\mathscr{P}(k,l) = \sum_{x=l+1}^{k} \mathscr{S}(x, 1^{n-x}) + \sum_{x=2}^{k} \sum_{\substack{y=2\\ y\le x}}^{l+1} \sum_{\substack{z=L-x\\ 0\le z\le L'-y}}^{K-x} \mathscr{S}\big(x, y, 2^z, 1^{n-x-y-2z}\big),$$

where

$$L = \begin{cases} l & \text{if } 2|y \\ l+1 & \text{otherwise} \end{cases}, \qquad L' = \begin{cases} l+1 & \text{if } 2|y \\ l & \text{otherwise} \end{cases},$$

$$K = \begin{cases} k-1 & \text{if } 2|x \\ k & \text{otherwise} \end{cases},$$

and in the multiple sum

(i) *if* $x \not\equiv y \pmod 2$ *then* $z \equiv x \pmod 2$

(ii) *if* $x = y$ *then* y *is even*

(iii) *if* $n - x - y - 2z = 0$ *then* z *is even.*

Proof. By Theorem 7.1(ii), $2!\mathscr{P}(k,l) = \mathscr{P}(k, l+1, 1)\downarrow 0^2$. As in the previous proof let $S = \mathscr{S}(a, b, 2^c, 1^{n+2-a-b-2c})$ be a general term in $\mathscr{P}(k, l+1, 1)$; then $S\downarrow 0^2 \subseteq \mathscr{P}(k,l)$ and referring to (7.4) we see that

$$\begin{aligned} S\downarrow 0^2 = 2\big(&\mathscr{S}(a, b, 2^{c-1}, 1^{n+2-a-b-2c}) + \mathscr{S}(a, b-1, 2^c, 1^{n+1-a-b-2c}) \\ &+\mathscr{S}(a, b-1, 2^{c-1}, 1^{n+3-a-b-2c}) + \mathscr{S}(a-1, b, 2^c, 1^{n+1-a-b-2c}) \\ &+\mathscr{S}(a-1, b, 2^{c-1}, 1^{n+3-a-b-2c}) \\ &+\mathscr{S}(a-1, b-1, 2^c, 1^{n+2-a-b-2c})\big). \end{aligned}$$

Considering each term to be written in the form $\mathscr{S}(\mu)$ where $\mu = (x, y, 2^z, 1^{n-x-y-2z})$ we see that if $x \not\equiv y \pmod 2$ then either two nodes were removed from the first two columns of S or two nodes were removed from the first two rows of S so $z \equiv x \pmod 2$. The only other restrictions occur when $x = y$ or $n - x - y - 2z = 0$. In the first case y must be even since this can happen only if the last node in the second row of μ has 2-residue 0. Similarly, in the second case z is even.

Checking the upper and lower bounds on x, y, and z requires more effort this time; however, it is not difficult. For example, for our ‘‘generic’’ $S \subseteq \mathscr{P}(k, l+1, 1)$ we know that $c \le k - a$. So, if we remove a node from the first row of S but not from the second column (so $x = a - 1$ and $z = c$), then x is even and $z \le k - x - 1$. Similarly, if x is odd then $z \le k - x$; this justifies our definition of K. ∎

(7.7) THEOREM. *Suppose that $\lambda = (k, l) \equiv (0, 0) \pmod 2$ is a 2-regular partition of n. Then*

$$\mathscr{P}(k,l) = \sum_{x=l+1}^{k} \mathscr{S}(x, 1^{n-x}) + \sum_{x=2}^{k} \sum_{\substack{y=2 \\ y \le x}}^{l+1} \sum_{\substack{z=l-x \\ 0 \le z \le l+1-y}}^{k-x} d_{xyz} \mathscr{S}(x, y, 2^z, 1^{n-x-y-2z}),$$

where in the multiple sum if $x \not\equiv y \pmod 2$ then $z \equiv x \pmod 2$ and

$$d_{xyz} = \begin{cases} 1, & \text{if } z = k - x,\ z = l + 1 - y, \text{ or } z = l - x \\ 2, & \text{otherwise}. \end{cases}$$

Proof. By Theorem 7.1(iv) $\mathscr{P}(k, l) = \mathscr{P}(k-1, l, 1)\downarrow 0 \uparrow 1$. As above let $S = \mathscr{S}(a, b, 2^c, 1^{n-a-b-2c})$ be a general term in $\mathscr{P}(k-1, l, 1)$; then $S \downarrow 0 \uparrow 1 \subseteq \mathscr{P}(k, l)$ and referring to (7.4) we see that

$$\begin{aligned} S\downarrow 0 \uparrow 1 = {} & \mathscr{S}(a+1, b, 2^c, 1^{n-1-a-b-2c}) + \mathscr{S}(a+1, b, 2^{c-1}, 1^{n+1-a-b-2c}) \\ & + \mathscr{S}(a+1, b-1, 2^c, 1^{n-a-b-2c}) \\ & + \mathscr{S}(a, b+1, 2^c, 1^{n-1-a-b-2c}) \\ & + \mathscr{S}(a, b+1, 2^{c-1}, 1^{n+1-a-b-2c}) \\ & + \mathscr{S}(a, b, 2^{c+1}, 1^{n-2-a-b-2c}) + \mathscr{S}(a, b, 2^{c-1}, 1^{n+2-a-b-2c}) \\ & + \mathscr{S}(a, b-1, 2^{c+1}, 1^{n-1-a-b-2c}) \\ & + \mathscr{S}(a, b-1, 2^c, 1^{n+1-a-b-2c}) \\ & + \mathscr{S}(a-1, b+1, 2^c, 1^{n-a-b-2c}) \\ & + \mathscr{S}(a-1, b, 2^{c+1}, 1^{n-1-a-b-2c}) \\ & + \mathscr{S}(a-1, b, 2^c, 1^{n+1-a-b-2c}). \end{aligned}$$

Again consider each term to be written in the form $\mathscr{S}(x, y, 2^z, 1^{n-x-y-2z})$. Since we have added and removed nodes of different parities we get parity restrictions upon x, y, and z if and only if these nodes affected just the first two rows of S or just its first two columns. This happens if and only if $x \not\equiv y \pmod 2$ and it is easy to see that this forces $z \equiv x \pmod 2$. Once more we leave the hook partitions to the reader.

It is straightforward to check that the upper and lower bounds on x, y, z are as stated, so it remains to verify the multiplicities of these Specht modules. So suppose that $\mu = (x, y, 2^z, 1^{n-x-y-2z})$ is a partition of n such that $\mathscr{S}(\mu) \subseteq \mathscr{P}(k,l)$ and consider the nodes at the end of the first two rows and first two columns in the residue diagram of μ. Then, because $\mathscr{S}(\mu) \subseteq \mathscr{P}(k+1,l,1)\downarrow 0 \uparrow 1$, exactly two of these nodes have 2-residue 0 and two have 2-residue 1. Therefore, removing one of the nodes of 2-residue 1 and "covering" the other node which has 2-residue 1 gives a partition in $\mathscr{P}(k-1,l,1)$. Consequently, $d_{\mu\lambda}$ is at most 2, and $d_{\mu\lambda} = 1$ if and only if it is not possible to remove and "cover" the two nodes with 2-residue 1 in either order (note that it is always possible to do this in at least one way). If z is one of its extreme values $k-x$, $l+1-y$, or $l-x$ then a little thought shows that $d_{\mu\lambda} = 1 = d_{xyz}$ as claimed. The only other time when $d_{\mu\lambda} = 1$ is when $x = y$ or $n - x - y - 2z = 0$. In the first case since $l - x \le z \le l + 1 - y$ we must have $z = l - x$ or $l + 1 - y$. In the second case, the Littlewood–Richardson rule again shows that z must be one of $k - x$, $l + 1 - y$, or $l - x$. ∎

8. APPLICATIONS TO 2-REGULAR PARTITIONS WITH 3 PARTS

Constructing the indecomposable $\mathscr{H}$-modules for 2-regular partitions of length 3 requires considerably more effort than the 2-part case; in fact there is one case where we do not know the complete answer. For 3-part partitions there are essentially eight cases, again corresponding to the possible choices for $\lambda \pmod 2$.

By Theorem 5.7 and Theorem 6.6 we know the indecomposable modules for alternating partitions of the form $(\lambda_1, \lambda_2, \lambda_3, 2, 1)$ and $(\mu_1, \mu_2, \mu_3, 1)$. With this as our starting point we show how to construct the indecomposables for 2-regular partitions of length 3 (with one exception).

(8.1) THEOREM. *Let $\lambda = (k,l,m)$ be a 2-regular partition with $m > 0$.*

(i) *If $\lambda \equiv (0,1,0) \pmod 2$ then $\mathscr{P}(\lambda) = \mathscr{P}(k,l,m,1)\downarrow 1$.*

(ii) *If $\lambda \equiv (0,1,1) \pmod 2$ then $2\mathscr{P}(\lambda) = \mathscr{P}(k,l,m+1,1)\downarrow 1^2$.*

(iii) *If $\lambda \equiv (0,0,1) \pmod 2$ then $3!\mathscr{P}(\lambda) = \mathscr{P}(k,l+1,m+1,1)\downarrow 1^3$.*

(iv) *If $\lambda \equiv (1,0,1) \pmod 2$ and $m > 1$ then*

$$\mathscr{P}(k,l,m,2,1)\downarrow 0^2 1 = 2\mathscr{P}(\lambda) \oplus 2\mathscr{P}(k,l,m-2,2) \oplus 2\mathscr{P}(k,l-2,m,2) \oplus 2\mathscr{P}(k-2,l,m,2).$$

(v) *If $\lambda \equiv (1,0,0) \pmod 2$ then $\mathscr{P}(\lambda) = \mathscr{P}(k,l,m+1)\downarrow 0$.*

(vi) *If $\lambda \equiv (1,1,0) \pmod 2$ then*

$$\mathscr{P}(k,l+1,m+1)\downarrow 0^2 = 2\mathscr{P}(\lambda) \oplus 2\mathscr{P}(k,l+1,m-2,1) \oplus 2\mathscr{P}(k,l-1,m,1) \oplus 2\mathscr{P}(k-2,l+1,m,1)$$

(vii) *If* $\lambda = (k, l, m) \equiv (1, 1, 1) \pmod 2$ *then*

$$\mathscr{P}(k-1, l, m)\uparrow 0 = \mathscr{P}(\lambda) \oplus \mathscr{P}(k-2, l-1, m-1, 3, 1).$$

(8.2) *Remarks*. (i) If $\lambda \equiv (1, 0, 1) \pmod 2$ and $m = 1$ then the indecomposable $\mathscr{P}(\lambda)$ is given by Theorem 5.7. In particular, λ is the unique 2-regular partition in $\mathscr{P}(\lambda)$.

(ii) The only case not covered by the theorem or the previous remark is the case where all of the parts of λ are even. In Theorem 8.18 we give a partial answer in this case.

(iii) The theorem gives an algorithm for calculating all the decomposition numbers $d_{\mu\lambda}$ where λ is a 2-regular partition of length 3 with at least one odd part. Note that in (iv) and (vi) where $\mathscr{P}(\lambda)$ is constructed as a proper submodule of a projective $\mathscr{H}$-module P the other indecomposable summands of P are known by Theorem 6.6. Explicitly, if $(k, l, m) \equiv (1, 0, 1)$ $\pmod 2$ as in (iv) then

$$\begin{aligned}
\mathscr{P}(k, l, m-2, 2) &= \mathscr{P}(k, l, m-2, 2, 1)\downarrow 0\\
\mathscr{P}(k, l-2, m, 2) &= \mathscr{P}(k, l-2, m, 2, 1)\downarrow 0\\
\mathscr{P}(k-2, l, m, 2) &= \mathscr{P}(k-2, l, m, 2, 1)\downarrow 0
\end{aligned}$$

and if $(k, l, m) \equiv (1, 1, 0) \pmod 2$ as in (vi) then (cf. Example 6.7),

$$\begin{aligned}
3!\mathscr{P}(k, l+1, m-2, 1) &= \mathscr{P}(k, l+1, m-1, 2, 1)\downarrow 0^3\\
3!\mathscr{P}(k, l-1, m, 1) &= \mathscr{P}(k, l-1, m+1, 2, 1)\downarrow 0^3\\
3!\mathscr{P}(k-2, l+1, m, 1) &= \mathscr{P}(k-2, l+1, m+1, 2, 1)\downarrow 0^3.
\end{aligned}$$

In (vii) we need to know the indecomposable $\mathscr{P}(k-2, l-1, m-1, 3, 1)$ where k, l, and m are all odd; this module is constructed in Lemma 8.16. It is also true in this case that $2\mathscr{P}(k, l, m) = \mathscr{P}(k, l, m+1)\downarrow 1$; however, the proof of this is exceedingly messy.

Our proof of Theorem 8.1 is long and tedious; the reader uninterested in the details is invited to skip to Corollary 8.21. The strategy we employ is our usual one: we first identify all of the Specht modules corresponding to 2-regular partitions (together with their multiplicities) in each of the projectives obtained by the restrictions given above and then we show that the indecomposables corresponding to these 2-regulars cannot split off.

Our main tool in the second part of the argument is the Scattering Theorem (6.4) which in most cases allows us either to calculate the relevant decomposition numbers directly or to show that the corresponding indecomposable contains a partition which we know is not in the projective P we are considering. For example, if μ is a 2-regular partition with fourth part 3 and one of the first three rows ends with a node of residue 0 then the Scattering Theorem shows that $d_{\mu^{(i\downarrow)(4\uparrow)}\mu} = 1$ for some $i \in [3]$; so $\mathscr{P}(\mu)$ has girth at least 4. However, all of the projectives P above have girth at most 3 (by (4.3) and Corollary 4.6), so this shows that $\mathscr{P}(\mu) \nsubseteq P$.

Listing the 2-regular partitions which can appear in these projectives is more time consuming; however, here we are assisted by the observation made above that all of the modules considered have girth at most 3. Consequently we only need consider 2-regular partitions of the form

$$\begin{array}{ll} (\mu_1, \mu_2, \mu_3) & (\mu_1, \mu_2, \mu_3, 1) \\ (\mu_1, \mu_2, \mu_3, 2) & (\mu_1, \mu_2, \mu_3, 3) \\ (\mu_1, \mu_2, \mu_3, 2, 1) & (\mu_1, \mu_2, \mu_3, 3, 1) \\ (\mu_1, \mu_2, \mu_3, 3, 2) & (\mu_1, \mu_2, \mu_3, 3, 2, 1) \end{array} \tag{8.3}$$

Most of the 2-regular partitions which can appear come from restricting 2-regular partitions "higher up." In identifying the 2-regular partitions which come from restricting 2-singular partitions the following two lemmas will be useful.

(8.4) LEMMA. *Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 2, 1)$ *be an alternating partition and suppose that* $d_{\mu\lambda} > 0$. *Then*

(i) *If* $\mu = (\mu_1, \mu_2, \mu_3, 3^2, 2, 1)$ *then* $\mu = (\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 2, 3^2, 2, 1)$.

(ii) *If* $\mu = (\mu_1, \mu_2, \mu_3, 2^3, 1)$ *then* μ *is one of the partitions* $(\lambda_1, \lambda_2 - 2, \lambda_3 - 2, 2^3, 1)$, $(\lambda_1 - 2, \lambda_2, \lambda_3 - 2, 2^3, 1)$, *or* $(\lambda_1 - 2, \lambda_2 - 2, \lambda_3, 2^3, 1)$.

(iii) *If* $\mu = (\mu_1, \mu_2, \mu_3, 2, 1^3)$ *then* μ *is one of the partitions* $(\lambda_1, \lambda_2, \lambda_3 - 2, 2, 1^3)$, $(\lambda_1, \lambda_2 - 2, \lambda_3, 2, 1^3)$, *or* $(\lambda_1 - 2, \lambda_2, \lambda_3, 2, 1^3)$.

(iv) *If* $\mu = (\mu_1, \mu_2, \mu_3, 3^2, 2, 1^3)$ *then* μ *is one of the partitions* $(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 4, 3^2, 2, 1^3)$, $(\lambda_1 - 2, \lambda_2 - 4, \lambda_3 - 2, 3^2, 2, 1^3)$, *or* $(\lambda_1 - 4, \lambda_2 - 2, \lambda_3 - 2, 3^2, 2, 1^3)$.

Moreover, in all cases $d_{\mu\lambda} = 1$.

Proof. Part (i) is proved in the course of Theorem 6.6. The remaining three cases are proved similarly using Theorem 5.7 and the Littlewood–Richardson rule. ∎

(8.5) LEMMA. *Suppose that* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1)$ *is an alternating partition. Then*

$$5!\mathscr{P}(\lambda) = \mathscr{P}(\lambda_1 + 1, \lambda_2 + 1, \lambda_3 + 1, 2, 1)\downarrow 0^5$$

and the following partitions, each with multiplicity 1, *are contained in* $\mathscr{P}(\lambda)$:

$$
\begin{array}{ll}
(\lambda_1, \lambda_2, \lambda_3, 1) & \\
(\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 2, 3, 2, 1), & (\lambda_1 - 2, \lambda_2 - 1, \lambda_3 - 2, 3, 2, 1) \\
(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 1, 3, 2, 1) & \left(\lambda_1, \lambda_2 - 2, \lambda_3 - 2, 2^2, 1\right), \\
\left(\lambda_1 - 2, \lambda_2, \lambda_3 - 2, 2^2, 1\right) & \left(\lambda_1 - 2, \lambda_2 - 2, \lambda_3, 2^2, 1\right)
\end{array}
$$

This list includes all of the 2-*regular partitions in* $\mathscr{P}(\lambda)$. *Moreover, if* $\mathscr{S}(\mu) \subseteq \mathscr{P}(\lambda)$ *and* $\mathscr{S}(\mu)\downarrow 1^a$ *contains a* 2-*regular partition for some* $a \geq 0$, *then* μ *is in this list*.

Proof. By Theorem 6.6, $5!\mathscr{P}(\lambda) = \mathscr{P}(\lambda_1 + 1, \lambda_2 + 1, \lambda_3 + 1, 2, 1)\downarrow 0^5$, and this result also identifies the regular partitions in $\mathscr{P}(\lambda)$.

Assume now that $\mathscr{S}(\nu) \subseteq \mathscr{P}(\lambda_1 + 1, \lambda_2 + 1, \lambda_3 + 1, 2, 1)$. Then ν can be written in the form $\nu = (\nu^h; \nu^v)_5$ by Theorem 5.7. Suppose that $\mathscr{S}(\nu)\downarrow 0^5$ contains a 2-regular partition. We find that the only possibilities for ν^v are $\nu^v = (0)$ or $\nu^v = (1^3)$. In the former case, $\mathscr{S}(\nu)\downarrow 0^5 = 5!\mathscr{S}(\lambda_1, \lambda_2, \lambda_3, 1)$. If $\nu^v = (1^3)$ then $\nu = (\nu_1, \nu_2, \nu_3, 3^2, 2, 1) = (\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1, 3^2, 2, 1)$ by Lemma 8.4(i), and $\mathscr{S}(\nu)\downarrow 0^5$ is one of $5!\mathscr{S}(\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 2, 3, 2, 1)$, $5!\mathscr{S}(\lambda_1 - 2, \lambda_2 - 1, \lambda_3 - 2, 3, 2, 1)$, $5!\mathscr{S}(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 1, 3, 2, 1)$, or $5!\mathscr{S}(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 2, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ where $\sigma = (3^2, 1), (3, 2^2)$ or $(3, 2, 1^2)$.

Next suppose that $\mathscr{S}(\nu)\downarrow 0^5 1^a$ contains a 2-regular partition for some $a \geq 0$. This time $\nu^v = (0), (1^2)$, or (1^3). If $\nu^v = (1^2)$ then $\nu = (\lambda_1 + 1, \lambda_2 - 1, \lambda_3 - 1, 2^3, 1)$, $(\lambda_1 - 1, \lambda_2 + 1, \lambda_3 - 1, 2^3, 1)$, or $(\lambda_1 - 1, \lambda_2 - 1, \lambda_3 + 1, 2^3, 1)$ by Lemma 8.4(ii), and when we 0-restrict these partitions five times we obtain the last three partitions in our list. Finally, note that the three 2-singular partitions of the form $(\lambda_1 - 2, \lambda_2 - 2, \lambda_3 - 2, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ do not 1-restrict to give 2-regular partitions. ∎

We now begin the proof of Theorem 8.1. In the course of proving the theorem we actually give more information than is provided in the statement of the theorem since we list the 2-regular partitions, together with their multiplicities, which appear in the indecomposables $\mathscr{P}(k, l, m)$. In fact, we do slightly more than this because we list all of the partitions in $\mathscr{P}(k, l, m)$ which have one of the types given in (8.3), and some of these partitions could be 2-singular. It may also happen that for some choices of k, l, and m some of the "partitions" in our lists have negative or increasing parts; by agreement we ignore such partitions. Most of the partitions in our lists will also have some of their parts underlined by solid or broken lines; this is a reference to the Scattering Theorem which will be explained in the course of the proofs.

(8.6). *Suppose that* $\lambda = (k, l, m) \equiv (0, 1, 0) \pmod 2$. *Then*

$$\mathscr{P}(\lambda) = \mathscr{P}(k, l, m, 1)\downarrow 1,$$

and the following partitions, each with multiplicity 1, *are contained in* $\mathscr{P}(\lambda)$:

$$\begin{array}{ll}
(k, l, m) \quad (k, l, \underline{m-1}, \dashuline{1}) & (k, \underline{l-1}, m, \dashuline{1}) \\
(\underline{k-1}, l, m, \dashuline{1}) & (\dashuline{k-1}, l-2, m-2, \underline{3}, 2) \\
(k-2, l-2, \dashuline{m-1}, \underline{3}, 2) & (k-2, \dashuline{l-1}, m-2, \underline{3}, 2) \\
(k-1, l-2, \dashuline{m-3}, \underline{3}, 2, 1) & (k-1, \dashuline{l-3}, m-2, \underline{3}, 2, 1) \\
(k-2, l-1, \dashuline{m-3}, \underline{3}, 2, 1) & (k-3, \dashuline{l-1}, m-2, \underline{3}, 2, 1) \\
(k-2, l-3, \dashuline{m-1}, \underline{3}, 2, 1) & (k-3, l-2, \dashuline{m-1}, \underline{3}, 2, 1).
\end{array}$$

Moreover, all of the 2*-regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list.*

Proof. Let $P = \mathscr{P}(k, l, m, 1)\downarrow 1$ and note that we know how to construct $\mathscr{P}(k, l, m, 1)$ by Theorem 6.6. We must show that $P = \mathscr{P}(\lambda)$. It is straightforward to check that 1-restricting the partitions listed in Lemma 8.5 gives precisely the list of partitions given above, all with multiplicity 1, together with some 2-singular partitions. So we need to show that $d_{\mu\lambda} = 1$ for all of the 2-regular partitions μ in this list. This brings us to the meaning of solid and broken lines in (8.6).

First, consider the partition $\mu = (k, l, \underline{m-1}, \dashuline{1})$. Now $\mu = \lambda^{(3\downarrow)(4\uparrow)}$ so $d_{\mu\lambda} = 1$ by the Scattering Theorem. Similarly, whenever a partition μ is listed with its ith part underlined with a solid line and its jth part underlined with a broken line and $j > i$ then it is the case that $\mu = \lambda^{(i\downarrow)(j\uparrow)}$, and $d_{\mu\lambda} = 1$ by the scattering Theorem (6.4).

Now consider $\mu = (\dashuline{k-1}, l-2, m-2, \underline{3}, 2)$. In this case a broken line appears before the solid line which is to indicate that we should consider the partition $\nu = \mu^{(1\downarrow)(4\uparrow)} = (k-2, l-2, m-2, 4, 2)$. Now $\mathscr{S}(\nu) \not\subseteq P$ since P has girth at most 3 by (4.3) and Corollary 4.6. Therefore $\mathscr{P}(\mu) \not\subseteq P$ since $d_{\nu\mu} = 1$ by the Scattering Theorem. The same argument (moving the node as indicated) shows that $\mathscr{P}(\mu) \not\subseteq P$ for all the remaining partitions in the list. Since $\mathscr{P}(\mu) \not\subseteq P$ for any 2-regular partition $\mu \neq \lambda$ it follows that $P = \mathscr{P}(\lambda)$ as claimed. ∎

We should remark that it is necessary to check whether or not the partitions $\nu = \mu^{(i\downarrow)(j\uparrow)}$ obtained above are non-zero. For example, in the case where $\mu = (k-1, l-2, m-2, 3, 2)$ if $m = 5$ then $\nu = \mu^{(1\downarrow)(4\uparrow)} = 0$ by our conventions. Since m is even in (8.6) this is not a problem; however, even if m could be odd our argument would still be valid for μ because when $m = 5$ the partition μ is 2-singular and so there would be no need to consider it (and $\mu = 0$ if $m < 5$). Hereafter we leave such details to the intrepid reader.

(8.7). *Suppose that* $\lambda = (k, l, m) \equiv (0, 1, 1) \pmod 2$ *is* 2-*regular*. *Then*

$$2\mathscr{P}(\lambda) = \mathscr{P}(k, l, m+1, 1)\downarrow 1^2,$$

and the following partitions, each with multiplicity 1, *are contained in* $\mathscr{P}(\lambda)$:

$$\begin{array}{ll}
(k, l, m) & (k, \underline{l-1}, \underline{m+1}) \\
(\underline{k-1}, \underline{m+1}) & (k, \underline{l-1}, m, \underline{1}) \\
(\underline{k-1}, l, m, \underline{1}) & (\underline{k-1}, \underline{l-1}, \underline{m+1}, \underline{1}) \\
(k, l-2, \underline{m-1}, \underline{2}, 1) & (k-2, l, \underline{m-1}, \underline{2}, 1) \\
(k-2, \underline{l-2}, m+1, \underline{2}, 1) & (\underline{k-1}, l-2, m-1, \underline{3}, 1) \\
(k-2, \underline{l-1}, m-1, \underline{3}, 1) & (k-2, l-2, \underline{m}, \underline{3}, 1) \\
(k-1, l-2, \underline{m-2}, \underline{3}, 2) & (k-1, \underline{l-3}, m-1, \underline{3}, 2) \\
(k-2, l-1, \underline{m-2}, \underline{3}, 2) & (k-3, \underline{l-1}, m-1, \underline{3}, 2) \\
(k-2, l-3, \underline{m}, \underline{3}, 2) & (k-3, l-2, \underline{m}, \underline{3}, 2) \\
(k-1, l-3, \underline{m-2}, \underline{3}, 2, 1) & (k-3, \underline{l-1}, m-2, \underline{3}, 2, 1) \\
(k-3, l-3, \underline{m}, \underline{3}, 2, 1). &
\end{array}$$

Moreover, *all of the* 2-*regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list*.

Proof. Let $2P = \mathscr{P}(k, l, m+1, 1)\downarrow 1^2$ and note that $\mathscr{P}(k, l, m+1, 1)$ is known by Theorem 6.6. By 1^2-restricting the partitions in Lemma 8.5 we obtain all of the partitions which we have listed, together with some 2-singular partitions.

It remains to show that each of the partitions listed in (8.7) is contained in $\mathscr{P}(\lambda)$; by treating each of the partitions separately using the Scattering Theorem as described in (8.6) this is completely routine (when $\mu = (k-1, l-1, m+1, 1)$ let $\nu = \mu^{(12\downarrow)(34\uparrow)}$ and apply the Scattering Theorem). ∎

We do one more example of a calculation using the Scattering Theorem. If $\mu = (k-2, \underline{l-2}, m+1, \underline{2}, 1)$ then let $\nu = \mu^{(2\downarrow)(4\uparrow)}$ and note that $\mathscr{S}(\nu) \nsubseteq P$ since it is not listed in (8.7). However, $d_{\nu\mu} = 1$ by the Scattering Theorem, so $\mathscr{P}(\mu) \nsubseteq P$. Note also that $d_{\mu^{(3\downarrow)(4\uparrow)}\mu} = 1$; however, since $\mathscr{S}(\mu^{(3\downarrow)(4\uparrow)}) \subseteq P$ (this partition is in our list), this observation is not helpful.

(8.8). *Suppose that* $\lambda = (k, l, m) \equiv (0, 0, 1) \pmod 2$ *is* 2-*regular*. *Then*

$$3!\mathscr{P}(\lambda) = \mathscr{P}(k, l+1, m+1, 1)\downarrow 1^3,$$

and the following partitions, each with the stated multiplicity, are contained in $\mathscr{P}(\lambda)$:

$$
\begin{array}{rlrl}
 & (k, l, m) & & \\
 & (\underline{k-1}, \underline{l+1}, m) & & (\underline{k-1}, l, \underline{m+1}) \\
 & (\underline{k-1}, l, m, \underline{1}) & (2-\varepsilon_3) & (k-1, l-1, m-1, 2, 1) \\
 & (\underline{k}, l-2, m-1, 2, 1) & & (\underline{k}, l-1, m-2, 2, 1) \\
(2-\varepsilon_1-\varepsilon_3) & (\underline{k-2}, \underline{l}, \underline{m-1}, \underline{2}, 1) & & (\underline{k-2}, l-2, m+1, \underline{2}, 1) \\
 & (\underline{k-2}, l+1, m-2, \underline{2}, 1) & (2-\varepsilon_2) & (\underline{k-2}, \underline{l-1}, \underline{m}, \underline{2}, 1) \\
 & (k-3, \underline{l-1}, m+1, \underline{2}, 1) & & (k-3, l+1, \underline{m-1}, \underline{2}, 1) \\
 & (k-1, l-1, \underline{m-2}, \underline{3}, 1) & & (k-1, \underline{l-2}, m-1, \underline{3}, 1) \\
 & (k-2, l, \underline{m-2}, \underline{3}, 1) & & (k-2, l-2, \underline{m}, \underline{3}, 1) \\
 & (k-3, \underline{l}, m-1, \underline{3}, 1) & & (k-3, l-1, \underline{m}, \underline{3}, 1) \\
 & (k-1, l-2, \underline{m-2}, \underline{3}, 2) & & (k-3, l, \underline{m-2}, \underline{3}, 2) \\
 & (k-3, l-2, \underline{m}, \underline{3}, 2) & &
\end{array}
$$

where

$$
\varepsilon_1 = \begin{cases} 0 & \text{if } k > l+2 \\ 1 & \text{otherwise} \end{cases}, \qquad \varepsilon_2 = \begin{cases} 0 & \text{if } l > m+1 \\ 1 & \text{otherwise} \end{cases}, \qquad \text{and}
$$

$$
\varepsilon_3 = \begin{cases} 0 & \text{if } m > 3 \\ 1 & \text{otherwise} \end{cases}.
$$

Moreover, all of the 2*-regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list.*

(8.9) *Remark.* Note that the coefficients ε_i, $i \in [3]$, are 0 whenever the corresponding partition is 2-regular. Since the decomposition numbers of the 2-regular partitions determine $\mathscr{P}(\lambda)$ these coefficients can essentially be ignored. Note also that some of the other partitions listed above also have multiplicities depending upon the ε_i. For example, the multiplicity of $(k-3, l, m-2, 3, 2)$ in $\mathscr{P}(\lambda)$ is $1-\varepsilon_1$; however, when $\varepsilon_1 = 1$ this partition is 0 by our conventions.

Proof. Let $3!P = \mathscr{P}(k, l+1, m+1, 1)\downarrow 1^3$ and note that $5!\mathscr{P}(k, l+1, m+1, 1) = \mathscr{P}(k+1, l+2, m+2, 2, 1)\downarrow 0^5$ is known by Theorem 6.6. Restricting the partitions given in Lemma 8.5 gives the partitions listed in (8.8). Note that the partitions $(k-2, l+1, m-1, 2^2, 1)$ and $(k-2, l-1, m+1, 2^2, 1)$ in $\mathscr{P}(k, l+1, m+1, 1)$ are 0 when $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$ respectively; this accounts for the coefficients ε_i, $i \in [2]$, above (the other restrictions of these two partitions are automatically 0 when the relevant $\varepsilon_i = 1$). The ε_3's come from the first two partitions in Lemma 8.5 ending in $(3, 2, 1)$ being 0 when $m = 3$.

The decomposition numbers for the two partitions in the list which have only their first part underlined can be calculated directly using first row removal (Theorem 1.2(i)) and Theorem 7.1(iii). All but one of the remaining partitions are taken care of by the Scattering Theorem, although the partition $\mu = (\underline{k-2}, \underline{l}, \underline{m-1}, \underline{2}, 1)$ requires more explanation. We have indicated that the Scattering Theorem should be applied simultaneously to the first and second rows and to the third and fourth rows of μ; this is allowed because

$$d_{(k-3,l+1,m-2,3,1)(k-2,l,m-1,2,1)} = d_{(k-3,l+1)(k-2,l)} d_{(m-2,3,1)(m-1,2,1)} = 1,$$

by the general version of row and column removal (see [8, Theorem 6.18]).

This leaves the partition $\mu = (k-1, l-1, m-1, 2, 1)$ (a quick check reveals that the Scattering Theorem is of no help to us here because all of the composition factors of $\mathscr{P}(\mu)$ given by this result are contained in P). By induction on $|\lambda|$, (8.14) below, and Theorem 1.2 we can remove the first two columns from μ to see that

$$d_{(k-2,l-2,m-1,4,1)\mu} = d_{(k-4,l-4,m-3,2)(k-3,l-3,m-3)} = 1.$$

Consequently, $\mathscr{P}(\mu)$ has girth at least 4 and so $\mathscr{P}(\mu) \not\subseteq P$ and therefore $P = \mathscr{P}(\lambda)$ as claimed. ∎

(8.10). *Suppose that* $\lambda = (k, l, m) \equiv (1, 0, 1) \pmod 2$ *where* $m > 1$. *Then*

$$\mathscr{P}(k, l, m, 2, 1){\downarrow}0^2 1 = 2\mathscr{P}(k, l, m) \oplus 2\mathscr{P}(k, l, m-2, 2)$$
$$\oplus\, 2\mathscr{P}(k, l-2, m, 2) \oplus 2\mathscr{P}(k-2, l, m, 2),$$

and the following partitions, each with the multiplicity 1, *are contained in* $\mathscr{P}(\lambda)$:

(k, l, m)

$(\underline{k}, l-1, m-2, 2, 1)$ $\quad(\underline{k}, l-2, m-1, 2, 1)$ $\quad(\underline{k-1}, l, m-2, \underline{2}, 1)$

$(\underline{k-1}, l-2, m, \underline{2}, 1)$ $\quad(k-2, l, \underline{m-1}, \underline{2}, 1)$ $\quad(k-2, \underline{l-1}, m, \underline{2}, 1)$.

Moreover, all of the 2*-regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list.*

Note that the three indecomposables $\mathscr{P}(k, l, m-2, 2)$, $\mathscr{P}(k, l-2, m, 2)$, and $\mathscr{P}(k-2, l, m, 2)$ are known by Theorem 6.6; see Remark 8.2(iii).

Proof. Let $2P = \mathscr{P}(k, l, m, 2, 1){\downarrow}0^2 1$ and note that $\mathscr{P}(k, l, m, 2, 1)$ is known by Theorem 5.7. The 2-regular partitions in P come either from restricting the partition $(k, l, m, 2, 1)$ which gives the partitions (k, l, m) and

$(k, l, m-2, 2)$ $\quad(k, l-2, m, 2)$ $\quad(k-2, l, m, 2)$

$(k, l-1, m-2, 2, 1)$ $\quad(k, l-2, m-1, 2, 1)$ $\quad(k-1, l, m-2, 2, 1)$

$(k-1, l-2, m, 2, 1)$ $\quad(k-2, l, m-1, 2, 1)$ $\quad(k-2, l-1, m, 2, 1)$

or from restricting a partition of the form $\mu = (\mu_1, \mu_2, \mu_3, 2, 1^3)$ from $\mathscr{P}(k, l, m, 2, 1)$. In the latter case Lemma 8.4(iii) shows that μ is one of the partitions $(k, l, m-2, 2, 1^3)$, $(k, l-2, m, 2, 1^3)$, or $(k-2, l, m, 2, 1^3)$; restricting these partitions gives the 2-regulars

$$
(8.11)\quad
\begin{array}{ll}
(k,l,m-3,2,1) & (k,l-1,m-2,2,1) \\
(k-1,l,m-2,2,1) & (l-\varepsilon_2)(k,l-2,m-1,2,1) \\
(k,l-3,m,2,1) & (k-1,l-2,m,2,1) \\
(k-2,l,m-1,2,1) & (1-\varepsilon_1)(k-2,l-1,m,2,1) \\
(k-3,l,m,2,1) &
\end{array}
$$

where ε_1 and ε_2 are as in (8.8).

Now $\mathscr{P}(\lambda)$ contains no 4-part partitions by Corollary 6.3(i), so

$$d_{(k,l,m-2,2)\lambda} = d_{(k,l-2,m,2)\lambda} = d_{(k-2,l,m,2)\lambda} = 0.$$

Next note that if $\mu \neq \lambda$ and μ is one of the partitions listed in (8.11) then $\mathscr{P}(\mu)$ is not contained in P by the Scattering Theorem. Also, by Theorem 6.6 (cf. Remark 8.2(iii)), the 2-regular partitions in the indecomposable $\mathscr{P}(k,l,m-2,2)$ are $(k,l,m-2,2)$ and the first three partitions in (8.11). Similarly, the 2-regular partitions in $\mathscr{P}(k,l-2,m,2)$ are $(k,l-2,m,2)$ and the fourth, fifth, and sixth partitions of (8.11), and the 2-regular partitions in $\mathscr{P}(k-2,l,m,2)$ are $(k-2,l,m,2)$ and the remaining partitions in (8.11). It follows that

$$P = \mathscr{P}(k,l,m-2,2) \oplus \mathscr{P}(k,l-2,m,2) \oplus \mathscr{P}(k-2,l,m,2) \oplus \mathscr{P}(\lambda),$$

and the proof is complete. ∎

For future reference we next investigate the case where $(k,l,m) \equiv (1,0,1) \pmod 2$ a little further.

(8.12) LEMMA. *Suppose that* $\lambda = (k,l,m) \equiv (1,0,1) \pmod 2$ *where* $m > 1$. *Then the following partitions, each with multiplicity* 1, *are contained in* $\mathscr{P}(\lambda)$

$$
\begin{array}{ll}
(k,l,m) & \\
(k,l,m-2,1^2) & (k,l-2,m,1^2) \\
(k-2,l,m,1^2) & (k,l-1,m-2,2,1) \\
(k,l-2,m-1,2,1) & (k-1,l,m-2,2,1) \\
(k-1,l-2,m,2,1) & (k-2,l,m-1,2,1) \\
(k-2,l-1,m,2,1) & (k-2,l-2,m-2,3^2) \\
(k-2,l-2,m-4,3^2,1^2) & (k-2,l-4,m-2,3^2,1^2) \\
(k-4,l-2,m-2,3^2,1^2) & (k-2,l-2,m-4,3,2^2,1) \\
(k-2,l-4,m-2,3,2^2,1) & (k-4,l-2,m-2,3,2^2,1).
\end{array}
$$

Moreover, if $\mathscr{S}(\mu) \subseteq \mathscr{P}(\lambda)$ *and* $\mathscr{S}(\mu)\downarrow 0^a$, *where* $a = 1$ *or* 2, *contains a* 2-*regular partition, then* μ *is one of the partitions in this list*.

Proof. Assume that $\mathscr{S}(\nu) \subseteq \mathscr{P}(k, l, m, 2, 1)$ and that $\mathscr{S}(\nu)\downarrow 0^2 10^a$ contains a 2-regular partition for $a = 1$ or $a = 2$. By Theorem 5.7 ν is 2-quotient separated so $\nu = (\nu^h; \nu^v)_5$, and the possibilities for ν^v are (0), (1), (1^3), and $(2, 1^2)$. We now refer to Lemma 8.4 to find out what ν can be and calculate all of the restrictions $\mathscr{S}(\nu)\downarrow 0^2 1$. We obtain the following partitions (along with some partitions ρ for which $\mathscr{S}(\rho)\downarrow 0^a$ contains only 2-singular partitions for $a = 1$ and 2):

$$\begin{array}{lll}
(k, l, m) & & \\
(k, l, m-2, 2) & (k, l-2, m, 2) & (k-2, l, m, 2) \\
(k, l, m-2, 1^2) & (k, l-2, m, 1^2) & (k-2, l, m, 1^2) \\
(k, l-1, m-2, 2, 1) & (k, l-2, m-1, 2, 1) & (k-1, l, m-2, 2, 1) \\
(k-1, l-2, m, 2, 1) & (k-2, l, m-1, 2, 1) & (k-2, l-1, m, 2, 1)
\end{array}$$

from the case where $\nu = (k, l, m, 2, 1)$;

$$\begin{array}{lll}
(k, l, m-2, 1^2) & (k, l-2, m, 1^2) & (k-2, l, m, 1^2) \\
(k, l-1, m-2, 2, 1) & (k, l-2, m-1, 2, 1) & (k-1, l, m-2, 2, 1) \\
(k-1, l-2, m, 2, 1) & (k-2, l, m-1, 2, 1) & (k-2, l-1, m, 2, 1) \\
(k, l, m-3, 2, 1) & (k, l-3, m, 2, 1) & (k-3, l, m, 2, 1)
\end{array}$$

from the cases where $\nu = (\nu_1, \nu_2, \nu_3, 2, 1^3)$;

$$\begin{array}{ll}
(k-2, l-2, m-2, 3^2) & \\
(k-2, l-2, m-4, 3^2, 2) & (k-2, l-4, m-2, 3^2, 2) \\
(k-4, l-2, m-2, 3^2, 2) & (k-2, l-2, m-4, 3^2, 1^2) \\
(k-2, l-4, m-2, 3^2, 1^2) & (k-4, l-2, m-2, 3^2, 1^2) \\
(k-2, l-2, m-4, 3, 2^2, 1) & (k-2, l-4, m-2, 3, 2^2, 1) \\
(k-4, l-2, m-2, 3, 2^2, 1) &
\end{array}$$

from the cases where $\nu = (\nu_1, \nu_2, \nu_3, 3^2, 2, 1)$; and

$$\begin{array}{ll}
(k-2, l-2, m-4, 3^2, 1^2) & (k-2, l-4, m-2, 3^2, 1^2) \\
(k-4, l-2, m-2, 3^2, 1^2) & (k-2, l-2, m-4, 3, 2^2, 1) \\
(k-2, l-4, m-2, 3, 2^2, 1) & (k-4, l-2, m-2, 3, 2^2, 1)
\end{array}$$

from the cases where $\nu = (\nu_1, \nu_2, \nu_3, 3^2, 2, 1^3)$.

We next calculate the multiplicities in

$$\mathscr{P}(k, l, m-2, 2, 1)\downarrow 0 \oplus \mathscr{P}(k, l-2, m, 2, 1)\downarrow 0 \oplus \mathscr{P}(k-2, l, m, 2, 1)\downarrow 0$$

of all of the partitions we have found so far. We obtain

$$
\begin{array}{lll}
(k, l, m-2, 2) & (k, l-2, m, 2) & (k-2, l, m, 2) \\
(k, l, m-2, 1^2) & (k, l-2, m, 1^2) & (k-2, l, m, 1^2) \\
(k, l-1, m-2, 2, 1) & (k, l-2, m-1, 2, 1) & (k-1, l, m-2, 2, 1) \\
(k-1, l-2, m, 2, 1) & (k-2, l, m-1, 2, 1) & (k-2, l-1, m, 2, 1) \\
(k, l, m-3, 2, 1) & (k, l-3, m, 2, 1) & (k-3, l, m, 2, 1)
\end{array}
$$

from the restrictions $\mathscr{S}(\nu_1, \nu_2, \nu_3, 2, 1){\downarrow}0$; and

$$
\begin{array}{ll}
(k-2, l-2, m-4, 3^2, 2) & (k-2, l-4, m-2, 3^2, 2) \\
(k-4, l-2, m-2, 3^2, 2) & (k-2, l-2, m-4, 3^2, 1^2) \\
(k-2, l-4, m-2, 3^2, 1^2) & (k-4, l-2, m-2, 3^2, 1^2) \\
(k-2, l-2, m-4, 3, 2^2, 1) & (k-2, l-4, m-2, 3, 2^2, 1) \\
(k-4, l-2, m-2, 3, 2^2, 1) &
\end{array}
$$

from the restrictions $\mathscr{S}(\nu_1, \nu_2, \nu_3, 3^2, 2, 1){\downarrow}0$.

According to (8.10), by taking the difference between the two sets of partitions which we have found, we are left with the partitions in $\mathscr{P}(\lambda)$ of the sort we want. The list of partitions which remains is precisely that given in the statement of the lemma. ∎

(8.13). *Suppose that* $\lambda = (k, l, m) \equiv (1, 0, 0) \pmod 2$ *is* 2-*regular. Then*

$$\mathscr{P}(\lambda) = \mathscr{P}(k, l, m+1){\downarrow}0$$

and the following partitions, each with the stated multiplicity, are contained in $\mathscr{P}(\lambda)$:

$$
\begin{array}{ll}
(k, l, m) & (\underline{k}, l-1, m+1) \\
(\underline{k-1}, l, \dashuline{m+1}) & (\underline{k}, l, m-1, 1) \\
(\underline{k}, l-2, m+1, 1) & (\dashuline{k-2}, l, m+1, \underline{1}) \\
(\underline{k}, l-1, m-1, 2) & (\underline{k}, l-2, m, 2) \\
(\dashuline{k-1}, l, m-1, \underline{2}) & (k-2, l, \dashuline{m}, \underline{2}) \\
(\dashuline{k-1}, l-2, m+1, \underline{2}) & (k-2, \dashuline{l-1}, m+1, \underline{2}) \\
(\underline{k}, l-1, m-2, 2, 1) & (\underline{k}, l-3, m, 2, 1) \\
(k-1, l, \dashuline{m-2}, \underline{2}, 1) & 2(k-1, \dashuline{l-1}, m-1, \underline{2}, 1) \\
(2-\varepsilon_2)(k-1, l-2, \dashuline{m}, \underline{2}, 1) & (2-\varepsilon_1)(k-2, l-1, \dashuline{m}, \underline{2}, 1) \\
(k-3, l, \dashuline{m}, \underline{2}, 1) & (k-1, \dashuline{l-3}, m+1, \underline{2}, 1) \\
(k-3, \dashuline{l-1}, m+1, \underline{2}, 1) & (k-2, l-2, \dashuline{m-1}, \underline{3}, 2)
\end{array}
$$

where

$$\varepsilon_1 = \begin{cases} 0 & \text{if } k > l+1 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \varepsilon_2 = \begin{cases} 0 & \text{if } l > m+2 \\ 1 & \text{otherwise} \end{cases}.$$

Proof. Let $P = \mathscr{P}(k, l, m+1)\downarrow 0$ and note that $\mathscr{P}(k, l, m+1)$ is known by (8.10). By 0-restricting the partitions from Lemma 8.12 we obtain the partitions which we have listed, together with some 2-singular partitions. The Scattering Theorem now completes the proof. ∎

(8.14). *Suppose that* $\lambda = (k, l, m) \equiv (1,1,0) \pmod 2$ *is* 2-*regular*. *Then*

$$\mathscr{P}(k, l+1, m+1)\downarrow 0^2 = 2\mathscr{P}(\lambda) \oplus 2\mathscr{P}(k, l+1, m-2, 1) \oplus 2\mathscr{P}(k, l-1, m, 1) \oplus 2\mathscr{P}(k-2, l+1, m, 1)$$

and the following partitions, with the stated multiplicity are contained in $\mathscr{P}(\lambda)$:

$$\begin{array}{ll}
(k, l, m) & (\underline{k-1}, \underline{l+1}, m) \\
(\underline{k-1}, l, \underline{m+1}) & (\underline{k}, l, m-1, 1) \\
(k, l-1, m, 1) & (k-1, \underline{l+1}, m-1, \underline{1}) \\
(k-1, \underline{l-1}, m+1, \underline{1}) & (k-2, l+1, m, 1) \\
(k-2, l, \underline{m+1}, \underline{1}) & (k-1, \underline{l}, m-1, \underline{2}) \\
(k-1, l-1, \underline{m}, \underline{2}) & (k-2, l, \underline{m}, \underline{2}) \\
(k-1, l, \underline{m-2}, \underline{2}, 1) & (k-1, l-2, \underline{m}, \underline{2}, 1) \\
(k-3, l, \underline{m}, \underline{2}, 1) & (k-2, \underline{l-1}, m-2, \underline{3}, 2) \\
(k-2, l-2, \underline{m-1}, \underline{3}, 2) & (k-3, l-1, \underline{m-1}, \underline{3}, 2) \\
(k-2, l-1, \underline{m-3}, \underline{3}, 2, 1) & (k-2, l-3, \underline{m-1}, \underline{3}, 2, 1) \\
(k-4, l-1, \underline{m-1}, \underline{3}, 2, 1). &
\end{array}$$

Moreover, all of the 2-*regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list*.

Note that the three indecomposables $\mathscr{P}(k, l+1, m-2, 1)$, $\mathscr{P}(k, l-1, m, 1)$, and $\mathscr{P}(k-2, l+1, m, 1)$ are known by Example 6.7.

Proof. Let $2P = \mathscr{P}(k, l+1, m+1)\downarrow 0^2$ and note that $\mathscr{P}(k, l+1, m+1)$ is known by (8.10). First 0^2-restrict the partitions in Lemma 8.12 to

obtain the given partitions together with the partitions
(8.15)

$$\begin{array}{ll}
(k,l+1,m-2,1) & (k,l-1,m,1)\\
(k-2,l+1,m,1) & (k,l,m-1,1)\\
(k,l-2,m+1,1) & (k-2,l,m+1,1)\\
(k-1,l+1,m-1,1) & (k-1,l-1,m+1,1)\\
(k-3,l+1,m+1,1) & (k,l,m-2,2)\\
(k,l-2,m,2) & (k-2,l,m,2)\\
(k-1,l+1,m-2,2) & (k-1,l-1,m,2)\\
(k-3,l+1,m,2) & (k-1,l,m-1,2)\\
(k-1,l-2,m+1,2) & (k-3,l,m+1,2)\\
(k-1,l,m-2,2,1) & (k-1,l-2,m,2,1)\\
(k-3,l,m,2,1) & (k-2,l-1,m-3,3,2,1)\\
(k-2,l-3,m-1,3,2,1) & (k-4,l-1,m-1,3,2,1),
\end{array}$$

and also some 2-singular partitions. Note that if $m=2$ then the partitions whose position in the list is $\equiv 1 \bmod 3$ should be omitted; if $l=m+1$ then the partitions whose position in the list is $\equiv 2 \bmod 3$ should be omitted; and if $k=l+2$ then the partitions whose position in the list is $\equiv 0 \bmod 3$ should be omitted. Now that we have listed all of the 2-regular partitions in P we have to sort them into indecomposables.

First note that if $\mu \neq (k,l,m)$, $(k,l+1,m-2,1)$, $(k,l-1,m,1)$, or $(k-2,l+1,m,1)$ and μ is one of the partitions listed in (8.14) then the corresponding indecomposable $\mathscr{P}(\mu)$ is not contained in P by the Scattering Theorem. Next, $\mathscr{P}(k,l+1,m-2,1) \subseteq P$ since of all of the partitions we have listed $(k,l+1,m-2,1)$ appears earliest in the lexicographic order. By Example 6.7, the 2-regular partitions in $\mathscr{P}(k,l+1,m-2,1)$ are the partitions whose positions in the list (8.15) are $\equiv 1 \bmod 3$. Consequently P also contains $\mathscr{P}(k,l,m)$.

Next, first row removal shows that $d_{(k,l-1,m,1)(k,l,m)}=1$. However, $(k,l-1,m,1)$ appears with multiplicity 2 in P, so $\mathscr{P}(k,l-1,m,1) \subseteq P$. The 2-regular partitions in $\mathscr{P}(k,l-1,m,1)$ are given by Example 6.7, and their positions are $\equiv 2 \bmod 3$ in (8.15). (Note that if $l=m+1$ then $\mathscr{P}(k,l-1,m,1)=0$, since $(k,l-1,m,1)$ is 2-singular.)

To show that $d_{(k-2,l+1,m,1)(k,l,m)}=1$ we take an indirect route: it follows from (8.6) that $\mathscr{P}(k-1,l,m)\uparrow 0$ contains exactly one copy of $\mathscr{P}(\lambda)$; also $\mathscr{P}(k-1,l,m)\uparrow 0$ contains only one partition which induces to $(k-2,l+1,m,1)$, namely $(k-2,l,m,1)$. Hence, $d_{(k-2,l+1,m,1)(k,l,m)} \leq 1$. On the other hand, $d_{(k-3,l+1,m,2)(k-2,l+1,m,1)}=1$ and, by the Scattering Theorem, the Specht module $\mathscr{S}(k-3,l+1,m,2)$ appears with multiplicity 1 in P, whilst $(k-2,l+1,m,1)$ appears with multiplicity 2 in P.

Hence $d_{(k-2,l+1,m,1)(k,l,m)} = 1$ and P also contains $\mathscr{P}(k-2, l+1, m, 1)$. The 2-regular partitions in $\mathscr{P}(k-2, l+1, m, 1)$ are also given by Example 6.7 and their positions are $\equiv 0 \bmod 3$ in (8.15).

Since all of the 2-regular partitions in P have now been accounted for, it follows that

$$P = \mathscr{P}(k, l+1, m-2, 1) \oplus \mathscr{P}(k, l-1, m, 1)$$
$$\oplus \mathscr{P}(k-2, l+1, m, 1) \oplus \mathscr{P}(k, l, m),$$

and the proof is complete. ∎

Before we tackle the case where $\lambda \equiv (1,1,1) \pmod 2$ we need the lemma:

(8.16) LEMMA. *Suppose that* $\mu = (k-2, l-1, m-1, 3, 1) \equiv (1,0,0,1,1) \pmod 2$ *is* 2*-regular. Then* $2\mathscr{P}(\mu) = \mathscr{P}(k-2, l-1, m-1, 3, 2, 1)\downarrow 1^2$, *and the following partitions, each with multiplicity* 1, *are contained in* $\mathscr{P}(\mu)$:

$$\begin{array}{ll}
(k-2, l-1, m-1, 3, 1) & (k-2, \underline{l-2}, \underline{m}, 3, 1) \\
(\underline{k-3}, l-1, \underline{m}, 3, 1) & (k-2, l-1, \underline{m-2}, \underline{3}, 2) \\
(k-2, \underline{l-3}, m, \underline{3}, 2) & (\underline{k-4}, l-1, m, \underline{3}, 2) \\
(k-2, l-2, \underline{m-1}, \underline{4}, 1) & (k-3, l-1, \underline{m-1}, \underline{4}, 1) \\
(k-3, \underline{l-2}, m, \underline{4}, 1) & (k-2, \underline{l-2}, m-2, \underline{4}, 2) \\
(k-2, l-3, \underline{m-1}, \underline{4}, 2) & (\underline{k-3}, l-1, m-2, \underline{4}, 2) \\
(\underline{k-3}, l-3, m, \underline{4}, 2) & (k-4, l-1, \underline{m-1}, \underline{4}, 2) \\
(k-4, \underline{l-2}, m, \underline{4}, 2) & (k-2, l-3, \underline{m-2}, 4, \underline{2}, 1) \\
(k-4, l-1, \underline{m-2}, 4, \underline{2}, 1) & (k-4, l-3, \underline{m}, 4, \underline{2}, 1) \\
(k-3, \underline{l-2}, m-2, \underline{4}, 3) & (k-3, l-3, \underline{m-1}, \underline{4}, 3) \\
(k-4, l-2, \underline{m-1}, \underline{4}, 3) & (\underline{k-3}, l-3, m-2, \underline{4}, 3, 1) \\
(k-4, \underline{l-2}, m-2, \underline{4}, 3, 1) & (k-4, l-3, \underline{m-1}, \underline{4}, 3, 1) \\
(k-3, l-4, \underline{m-3}, \underline{4}, 3, 2, 1) & (k-5, l-2, \underline{m-3}, \underline{4}, 3, 2, 1) \\
(k-5, l-4, \underline{m-1}, \underline{4}, 3, 2, 1). &
\end{array}$$

Moreover, all of the 2*-regular partitions in* $\mathscr{P}(\mu)$ *appear in this list.*

Proof. The indecomposable module $\mathscr{P}(k-2, l-1, m-1, 3, 2, 1)$ is known by Theorem 6.6. The 2-regular partitions in $2\mathscr{P}(k-2, l-1, m-1, 3, 2, 1)\downarrow 1^2$ come from restricting the partitions

$$\begin{array}{ll}
(k-2, l-1, m-1, 3, 2, 1) & (k-2, l-2, m, 3, 2, 1) \\
(k-3, l-1, m, 3, 2, 1) & (k-2, l-2, m-1, 4, 2, 1) \\
(k-3, l-1, m-1, 4, 2, 1) & (k-3, l-2, m, 4, 2, 1) \\
(k-3, l-2, m-1, 4, 3, 1) & (k-3, l-4, m-3, 4, 3, 2^2, 1) \\
(k-5, l-2, m-3, 4, 3, 2^2, 1) & (k-5, l-4, m-1, 4, 3, 2^2, 1).
\end{array}$$

The Scattering Theorem now completes the proof. ∎

(8.17). *Suppose that* $\lambda = (k, l, m) \equiv (1, 1, 1) \pmod 2$ *is* 2-*regular. Then*

$$\mathscr{P}(k-1, l, m)\uparrow 0 = \mathscr{P}(\lambda) \oplus \mathscr{P}(k-2, l-1, m-1, 3, 1)$$

and following partitions, each with the stated multiplicity, are contained in $\mathscr{P}(\lambda)$:

(k, l, m)	$(k, l-1, m+1)$
$(k-1, l+1, m)$	$(k-1, l-1, m+2)$
$(k-2, l+1, m+1)$	$(k-2, l, m+2)$
$(k, l-1, m, 1)$	$(k-2, l+1, m, 1)$
$(k-2, l-1, m+2, 1)$	$(k-1, l-1, m, 2)$
$(k-2, l, m, 2)$	$(k-2, l-1, m+1, 2)$
$(k, l-2, m-1, 2, 1)$	$(k-1, l, m-2, 2, 1)$
$(k-1, l-3, m+1, 2, 1)$	$(k-3, l+1, m-1, 2, 1)$
$(k-3, l-2, m+2, 2, 1)$	$(k-4, l, m+1, 2, 1)$
$(2-\varepsilon_2)(k-1, l-2, m, 2, 1)$	$2(k-1, l-1, m-1, 2, 1)$
$(2-\varepsilon_1)(k-2, l, m-1, 2, 1)$	$2(k-2, l-2, m+1, 2, 1)$
$2(k-3, l, m, 2, 1)$	$(2-\varepsilon_1-\varepsilon_2)(k-3, l-1, m+1, 2, 1)$
$(k-2, l-1, m-1, 3, 1)$	$(k-2, l-2, m, 3, 1)$
$(k-3, l-1, m, 3, 1)$	$(k-2, l-1, m-2, 3, 2)$
$(k-2, l-3, m, 3, 2)$	$(k-4, l-1, m, 3, 2)$,

where

$$\varepsilon_i = \begin{cases} 0 & \textit{if } \lambda_i > \lambda_{i+1} + 2 \\ 1 & \textit{otherwise} \end{cases}.$$

Moreover, all of the 2-*regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list.*

Note that $\varepsilon_i = 0$ whenever the corresponding partition is 2-regular (see also Remark 8.9).

Proof. Let $P = \mathscr{P}(k-1, l, m)\uparrow 0$ and note that $\mathscr{P}(k-1, l, m)$ is known by (8.7); so $5!2P = \mathscr{P}(k, l+1, m+2, 2, 1)\downarrow 0^5 1^2 \uparrow 0$. Restricting and inducing the terms $\mathscr{S}(\mu)$ in $\mathscr{P}(k, l+1, m+2, 2, 1)$ gives the partitions listed in Lemma 8.12 and (8.17) together with some 2-singular partitions.

First consider the fourth, fifth, and sixth partitions in the list above. By using Theorem 1.2 to remove the first m columns from $\mu = (k-1, l-1, m+2)$ and λ we see that

$$d_{\mu\lambda} = d_{(k-1, l-1, m+2)(k, l, m)} = d_{(k-m-1, l-m-1, 2)(k-m, l-m)} = 1,$$

the last equality following by Theorem 7.1(iv) since $(k-m, l-m) \equiv (0, 0) \pmod 2$. A similar argument shows that $d_{\nu\lambda} = 1$ if $\nu = (k-2, l+1, m+1)$ or $(k-2, l, m+2)$.

Next consider the partition $\mu = (k-2, l-1, m-1, 3, 1)$ which appears with multiplicity 2 in P. In Scattering Theorem (6.4) and the argument above show that all of the partitions in P which dominate μ are contained in $\mathscr{P}(\lambda)$. Now, $\mathscr{P}(k, l, m+1)\!\downarrow\!1 \supseteq 2\mathscr{S}(k, l, m)$ and (k, l, m) is the first partitions which appears in $\mathscr{P}(k, l, m+1)\!\downarrow\!1$ by (8.14); hence $\mathscr{P}(k, l, m+1)\!\downarrow\!1 \supseteq 2\mathscr{P}(k, l, m)$. Also, we find that the only partitions ν in $\mathscr{P}(k, l, m+1)$ such that $\mathscr{S}(\nu)\!\downarrow\!1 \supseteq \mathscr{S}(\mu)$ are $(k-2, l-1, m-1, 3, 2)$ and $(k-2, l-1, m-1, 3, 1^2)$. Therefore $\mathscr{S}(\mu)$ appears with multiplicity 2 in $\mathscr{P}(k, l, m+1)\!\downarrow\!1$. It follows that $d_{\mu\lambda} \le 1$ and $\mathscr{P}(\mu) \subseteq P$. However, $2\mathscr{P}(\mu) \not\subseteq P$ by Lemma 8.16. Therefore $d_{\mu\lambda} = 1$, and it remains to show that the partitions in (8.17) which appear after μ in the lexicographic order are in $\mathscr{P}(\lambda)$. The Scattering Theorem now completes the proof. ∎

This completes the proof of Theorem 8.1.

There is one case not covered by Theorem 8.1; namely those 3-part partitions with three even parts. In this case we almost know the complete answer, however we are unable to determine (essentially) one decomposition number. Getting even this partial answer in this case requires substantially more effort than any of the cases considered in Theorem 8.1.

(8.18) THEOREM. *Suppose that* $\lambda = (k, l, m) \equiv (0, 0, 0) \pmod 2$ *is* 2*-regular. Then*

$$\mathscr{P}(k, l, m+1)\!\downarrow\!0 = \mathscr{P}(\lambda) \oplus \delta\mathscr{P}(k-2, l-1, m, 2, 1)$$

where $\delta = 0$ *if* $m = 2$ *and* $\delta \le 1$ *otherwise. In addition the following partitions, each with the stated multiplicity, appears in* $\mathscr{P}(\lambda)$:

$$\begin{array}{ll}
(k, l, m) & (\underline{k}, l-1, m+1) \\
(\underline{k-1}, \underline{l+1}, m) & (k-1, l-1, m+2) \\
(k-2, l+1, m+1) & (k-2, l, m+2) \\
(\underline{k}, l, m-1, 1) & (\underline{k}, l-2, m+1, 1) \\
(\underline{k-1}, \underline{l+1}, \underline{m-1}, \underline{1}) & (k-1, \underline{l-2}, \underline{m+2}, 1) \\
(\underline{k-3}, \underline{l+1}, m+1, 1) & (k-3, \underline{l}, \underline{m+2}, 1) \\
2(k-1, l, m, 1) & (2-\varepsilon_2)(k-1, l-1, m+1, 1) \\
(2-\varepsilon_1)(k-2, l, m+1, 1) & (\underline{k}, l-1, m-1, 2) \\
(\underline{k}, l-2, m, 2) & (\underline{k-2}, l+1, m-1, \underline{2}) \\
(k-2, \underline{l-2}, \underline{m+2}, 2) & (k-3, l+1, \underline{m}, \underline{2}) \\
(\underline{k-3}, \underline{l-1}, m+2, 2) & (2-\varepsilon_3)(k-1, \underline{l-1}, \underline{m}, 2)
\end{array}$$

$$
\begin{array}{ll}
a_{\varepsilon_1,\varepsilon_3}(\underline{k-2},\underline{l},\underline{m},\underline{2}) & (2-\varepsilon_2)(\underline{k-2},\underline{l-1},\underline{m+1},\underline{2}) \\
(\underline{k},l-1,m-2,2,1) & (\underline{k},l-3,m,2,1) \\
(k-2,l+1,\underline{m-2},\underline{2},1) & (k-2,\underline{l-3},\underline{m+2},\underline{2},\underline{1}) \\
(k-4,l+1,\underline{m},\underline{2},1) & (k-4,l-1,\underline{m+2},\underline{2},1) \\
b_{\varepsilon,\delta}(k-2,l-1,m,2,1) & (k-1,l-1,\underline{m-1},\underline{3}) \\
(k-1,\underline{l-2},m,\underline{3}) & (k-2,\underline{l},m-1,\underline{3}) \\
(k-2,l-2,\underline{m+1},\underline{3}) & (k-3,\underline{l},m,\underline{3}) \\
(k-3,l-1,\underline{m+1},\underline{3}) & (\underline{k-1},l-1,m-2,\underline{3},1) \\
(\underline{k-1},l-3,m,\underline{3},1) & (k-2,\underline{l},m-2,\underline{3},1) \\
(k-2,l-3,\underline{m+1},\underline{3},1) & (k-4,\underline{l},m,\underline{3},1) \\
(k-4,l-1,\underline{m+1},\underline{3},1) & (2-\delta)(k-2,l-1,\underline{m-1},\underline{3},1) \\
c_{\varepsilon_2,\delta}(k-2,\underline{l-2},m,\underline{3},1) & c_{\varepsilon_1,\delta}(\underline{k-3},l-1,m,\underline{3},1) \\
(k-1,\underline{l-2},m-2,\underline{3},2) & (k-1,l-3,\underline{m-1},\underline{3},2) \\
(k-3,\underline{l},m-2,\underline{3},2) & (k-3,l-3,\underline{m+1},\underline{3},2) \\
(k-4,l,\underline{m-1},\underline{3},2) & (k-4,l-2,\underline{m+1},\underline{3},2) \\
(2-\delta)(k-2,l-2,\underline{m-1},\underline{3},2) & c_{\varepsilon_1,\delta}(k-3,l-1,\underline{m-1},\underline{3},2) \\
c_{\varepsilon_2,\delta}(k-3,\underline{l-2},m,\underline{3},2) & (k-1,l-2,\underline{m-3},\underline{3},2,1) \\
(k-1,l-4,\underline{m-1},\underline{3},2,1) & (k-3,l,\underline{m-3},\underline{3},2,1) \\
(k-3,l-4,\underline{m+1},\underline{3},2,1) & (k-5,l,\underline{m-1},\underline{3},2,1) \\
(k-5,l-2,\underline{m+1},\underline{3},2,1) & b_{\varepsilon,\delta}(k-3,l-2,\underline{m-1},\underline{3},2,1)
\end{array}
$$

where $a_{\varepsilon_1,\varepsilon_3} = 2-\varepsilon_2-\varepsilon_3$, $b_{\varepsilon,\delta} = 6-\varepsilon_1-\varepsilon_2-2\varepsilon_3-\delta$, $c_{\varepsilon_i,\delta} = 2-\varepsilon_i-\delta$, *and*

$$\varepsilon_i = \begin{cases} 0 & \text{if } \lambda_i > \lambda_{i+1}+2 \\ 1 & \text{otherwise} \end{cases}$$

$(i \in [3])$. *Moreover, all of the* 2*-regular partitions in* $\mathscr{P}(\lambda)$ *appear in this list.*

Proof. Let $P = \mathscr{P}(k,l,m+1)\downarrow 0$. Then $5!3!P = \mathscr{P}(k+1,l+2,m+3,2,1)\downarrow 0^5 1^3 0$ and it is straightforward, although very tedious, to check that the 2-regular partitions in P come from restricting one of the partitions $(k+1,l+2,m+3,2,1)$, $(k+1,l+2,m+1,2,1^3)$, $(1-\varepsilon_2)(k+1,l,m+3,2,1^3)$, $(1-\varepsilon_1)(k-1,l+2,m+3,2,1^3)$, $(k+1,l,m+1,2^3,1)$, $(1-\varepsilon_1)(k-1,l+2,m+1,2^3,1)$, $(1-\varepsilon_2)(k-1,l,m+3,2^3,1)$, $(k-1,l,m+1,3^2,2,1)$, $(k-1,l,m-1,3^2,2,1^3)$, $(1-\varepsilon_2)(k-1,l-2,m+1,3^2,2,1^3)$, and $(1-\varepsilon_1)(k-3,l,m+1,3^2,2,1^3)$ from $\mathscr{P}(k+1,l+2,m+3,2,1)$ and that these partitions give all of the partitions listed in Theorem 8.18, with the stated multiplicities, together with some 2-singular partitions.

As usual, the Scattering Theorem takes care of most of the partitions in Theorem 8.18; however, this time there are seven exceptions. The decomposition numbers for the fourth, fifth, and sixth partitions of Theorem 8.18 can be calculated directly by removing the first column and then using (8.17).

The thirteenth, fourteenth, and fifteenth partitions also need to be considered. First consider $\mu = (k-2, l, m+1, 1)$; using first column removal we see that

$$d_{(k-4,l,m+2,2)(k-2,l,m+1,1)} = d_{(k-5,l-1,m+1,1)(k-3,l-1,m)} = 1$$

with the last equality following by (8.8). Since $\mathscr{S}(k-4, l, m+2, 2) \nsubseteq P$ it follows that $\mathscr{P}(\mu) \nsubseteq P$.

Now consider the partition $\mu = (k-1, l-1, m+1, 1)$. We may assume that $l > m+2$ since otherwise μ is 2-singular. Using first row removal followed by first column removal shows that

$$d_{(k-1,l-3,m+1,3)\mu} = d_{(l-3,m+1,3)(l-1,m+1,1)} = d_{(l-4,m,2)(l-2,m)} = 1,$$

where the last equality comes from Theorem 7.1(iv). Since $\mathscr{S}(k-1, l-3, m+1, 3) \nsubseteq P$ this shows that $d_{\mu\lambda} = 2$.

Next consider the partition $\mu = (k-1, l, m, 1)$. We leave it to the readers to convince themselves that it can happen that $d_{\nu\mu}\mathscr{S}(\nu) \subseteq P$ for all ν so we cannot use our standard trick. We know of no direct way to calculate $d_{\mu\lambda}$ so instead we use Theorem 1.2 to add a column to μ and λ. This shows that $d_{\mu\lambda} = d_{(k,l+1,m+1,2)(k+1,l+1,m+1,1)}$, and we compute this decomposition number instead. Let $\mu^* = (k, l+1, m+1, 2)$ and $\lambda^* = (k+1, l+1, m+1, 1)$ and consider the module P' where

$$(8.19) \qquad 6!4!2!P' = \mathscr{P}(k+1, l+2, m+3, 4, 3, 2, 1)\downarrow 0^6 1^4 0^2$$

(note that $\mathscr{P}(k+1, l+2, m+3, 4, 3, 2, 1)$ is known by Theorem 5.7). Then P' contains exactly one copy of $\mathscr{P}(\lambda^*)$. The only partitions in P' which dominate μ^* are contained in $\mathscr{S}(k+1, l+2, m+3, 4, 3, 2, 1)\downarrow 0^6 1^4 0^2$; in the lexicographic order the first ten of these partitions, together with their multiplicities in P', are

$$\begin{array}{ll}
(\underline{k+1}, l+1, m+1, 1) & (\underline{k+1}, l+1, m, 2) \\
(\underline{k+1}, l, m+2, 1) & (\underline{k+1}, l, m, 3) \\
(\underline{k+1}, l-1, m+2, 2) & (\underline{k+1}, l-1, m+1, 3) \\
(\underline{k+1}, l-1, m, 3, 1) & (\underline{k+1}, l-2, m-1, 3, 2, 1) \\
(\underline{k}, \underline{l+2}, m+1, 1) & (\underline{k}, \underline{l+2}, \underline{m}, \underline{2}) \\
2(k, l+1, m+1, 2). &
\end{array}$$

That $d_{\nu\lambda^*} = 1$ for the first eight partitions in this list can be shown using first row removal and (8.17). Next note that $\mathscr{S}(k-1, l+3, m+1, 1) \not\subseteq P'$ and $\mathscr{S}(k-1, l+3, m-1, 3) \not\subseteq P'$ since by (8.19) and Corollary 5.8 if $\mathscr{S}(\eta_1, \dots, \eta_k) \subseteq P'$ then $\eta_2 \le l+2$. Therefore, $\mathscr{P}(k-1, l+3, m+1, 1) \not\subseteq P'$ and $\mathscr{P}(k-1, l+3, m-1, 3) \not\subseteq P'$ so $d_{\mu^*\lambda^*} \le 2$. Now consider the partition $\nu = ((l+1)^2, 3^2, 1^{k-l+m-4})$. Since $\mathscr{S}(\nu)\uparrow 0^2 = 2$ it is clear that $\mathscr{S}(\nu) \not\subseteq P'$. However, we claim that $d_{\nu\mu^*} = 1$ which would show that $\mathscr{P}(\mu^*) \not\subseteq P'$ and hence that $d_{\mu^*\lambda^*} = d_{\mu\lambda} = 2$. That $d_{\nu\mu^*} = 1$ follows from the lemma:

(8.20) LEMMA. *Suppose that* $\mu^* = (k, l+1, m+1, 2) \equiv (0,1,1,0) \pmod 2$ *is* 2*-regular. Then*

$$\mathscr{P}(\mu^*) = \mathscr{P}(k, l+1, m+1, 2, 1)\downarrow 0$$

and the following partitions, each with multiplicity 1, *are contained in* $\mathscr{P}(\mu^*)$:

$$\begin{array}{ll}
(k, l+1, m+1, 2) & (k, \underline{l}, \underline{m+2}, 2) \\
(\underline{k-1}, m+1, \underline{m+2}, 2) & (k, l, \underline{m+1}, \underline{3}) \\
(k-1, l+1, \underline{m+1}, \underline{3}) & (k-1, \underline{l}, m+2, \underline{3}) \\
(\underline{k}, l+1, m, 2, 1) & (\underline{k}, l-1, m+2, 2, 1) \\
(\underline{k-2}, l+1, m+2, \underline{2}, 1) & (k, \underline{l}, m, \underline{3}, 1) \\
(k, l-1, \underline{m+1}, \underline{3}, 1) & (\underline{k-1}, l+1, m, \underline{3}, 1) \\
(\underline{k-1}, l-1, m+2, \underline{3}, 1) & (k-2, l+1, \underline{m+1}, \underline{3}, 1) \\
(k-2, \underline{l}, m+2, \underline{3}, 1) & (k-1, \underline{l}, m, \underline{3}, 2) \\
(k-1, l-1, \underline{m+1}, \underline{3}, 2) & (k-2, l, \underline{m+1}, \underline{3}, 2) \\
(k-1, l, \underline{m-1}, \underline{3}, 2, 1) & (k-1, l-2, \underline{m+1}, \underline{3}, 2, 1) \\
(k-3, l, \underline{m+1}, \underline{3}, 2, 1). &
\end{array}$$

Moreover, all of the 2*-regular partitions in* $\mathscr{P}(\mu^*)$ *appear in this list, and* $d_{((l+1)^2, 3^2, 1^{k-l+m-4})\mu^*} = 1$.

Proof. Since $2\mathscr{P}(k, l+1, m+1, 2, 1) = \mathscr{P}(k-1, l, m+1, 2, 1)\uparrow 1^2$ by Theorem 5.9(iii), the 2-regular partitions in $\mathscr{P}(k, l+1, m+1, 2, 1)$ are the 2-regular partitions amongst $(k, l+1, m+1, 2, 1)$ and

$$\begin{array}{ll}
(k, l, m+2, 2, 1) & (k, l, m+1, 3, 1) \\
(k-1, l+1, m+2, 2, 1) & (k-1, l+1, m+1, 3, 1) \\
(k-1, l, m+2, 3, 1) & (k-1, l, m+1, 3, 2).
\end{array}$$

Restricting these 2-regulars gives all but the last three 2-regulars in Lemma 8.20; these are the only 2-regular partitions which come from 0-restricting 2-singular partitions in $\mathscr{P}(k, l+1, m+1, 2, 1)$. That $\mathscr{P}(k, l+1, m+1, 2, 1)\downarrow 0 = \mathscr{P}(\mu^*)$ now follows by the Scattering Theorem. It is now an easy calculation using Theorem 5.9(i) and the Littlewood-Richardson rule to verify that $d_{((l+1)^2, 3^2, 1^{k-l+m-4})\mu^*} = 1$. ∎

To complete the proof of Theorem 8.18 we need to consider the partition $\mu = (k-2, l-1, m, 2, 1)$. If $m = 2$ then μ is 2-singular (and therefore $\delta = 0$); so we may assume that $m > 2$. Note that $3!\mathscr{P}(\mu) = \mathscr{P}(k-2, l-1, m, 3, 2, 1)\downarrow 1^3$ by Theorem 5.9. Now consider the partition $\nu = (k-2, m^2, 3, 1^{l-m-1})$. From our description of $\mathscr{P}(\mu)$ it is easy to see that $d_{\nu\mu} = 1$. On the other hand, a quick calculation shows that $\mathscr{S}(\nu)$ appears with multiplicity 1 in P. Therefore P contains at most one copy of $\mathscr{P}(\mu)$ (i.e., $\delta \leq 1$). The 2-regular partitions in $\mathscr{P}(\mu)$ are precisely those 2-regular partitions listed in Theorem 8.18 whose decomposition multiplicity in P depends upon δ, so the proof is complete. ∎

Although we have tried many different approaches we have been unable to distinguish between the two possible cases for the projective P considered in Theorem 8.18. The partition $(10, 6, 4)$ is the smallest partition for which the ambiguity arises; using the q-Schaper theorem of [10] one can show that $\delta = 0$ in Theorem 8.18 in this case. Indeed, in every case which we have checked, the q-Schaper theorem can be used to show that $\delta = 0$; unfortunately, the necessary computations for the general case are formidable.

Looking through the results from (8.6) to Theorem 8.18 we are able to express the Specht modules of all 2-regular partitions with 3 or 4 parts as a sum of irreducibles. For the 2-regular partitions of length 3 this can be done by simply checking through the lists of 2-regular partitions in each decomposable as given in Theorem 7.1 and Theorem 8.1. For 4-part partitions a little extra must be done. Note that the ambiguity in Theorem 8.18 is not a hindrance in this exercise because it does not affect partitions of length less than or equal to 4.

(8.21) COROLLARY. *Suppose that $\lambda = (k, l, m)$ is a 2-regular partition of length 3. Then*:

(i) *If* $\lambda \equiv (0,0,0) \pmod 2$ *or* $\lambda \equiv (1,1,1) \pmod 2$ *then* $\mathscr{S}(k, l, m) = \mathscr{D}(k, l, m) + \mathscr{D}(k, l+1, m-1) + \mathscr{D}(k+1, l-1, m) + \mathscr{D}(k+2, l, m-2)$.

(ii) *If* $\lambda \equiv (0,0,1) \pmod 2$ *or* $\lambda \equiv (1,1,0) \pmod 2$ *then* $\mathscr{S}(k, l, m) = \mathscr{D}(k, l, m) + \mathscr{D}(k+1, l-1, m) + \mathscr{D}(k+1, l, m-1) + \mathscr{D}(k+1, l+1, m-2)$.

(iii) *If* $\lambda \equiv (0,1,0) \pmod 2$ *or* $\lambda \equiv (1,0,1) \pmod 2$ *then* $\mathscr{S}(k,l,m) = \mathscr{D}(k,l,m)$.

(iv) *If* $\lambda \equiv (0,1,1) \pmod 2$ *or* $\lambda \equiv (1,0,0) \pmod 2$ *then* $\mathscr{S}(k,l,m) = \mathscr{D}(k,l,m) + \mathscr{D}(k,l+1,m-1) + \mathscr{D}(k+1,l,m-1) + \mathscr{D}(k+2,l-1,m-1)$.

(8.22) COROLLARY. *Suppose that* $\lambda = (k,l,m,p)$ *is a* 2*-regular partition of length* 4. *Then*:

(i) *If* $\lambda \equiv (0,0,0,0) \pmod 2$ *or* $\lambda \equiv (1,1,1,1) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2*-regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k,l,m+1,p-1)\\
(k,l+1,m-1,p) & (k,l+2,m,p-2)\\
(k+1,l-1,m,p) & (k+1,l-1,m+1,p-1)\\
(k+1,l,m,p-1) & 2(k+1,l+1,m-1,p-1)\\
(k+1,l+1,m,p-2) & (k+1,l+1,m+1,p-3)\\
(k+2,l,m-2,p) & (k+2,l,m-1,p-1)\\
2(k+2,l,m,p-2) & (k+2,l+2,m-2,p-2)\\
(k+3,l-1,m-1,p-1) & (k+3,l+1,m-1,p-3).
\end{array}$$

(ii) *If* $\lambda \equiv (0,0,0,1) \pmod 2$ *or* $\lambda \equiv (1,1,1,0) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2*-regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k,l+1,m-1,p)\\
(k,l+1,m,p-1) & (k,l+1,m+1,p-2)\\
(k+1,l-1,m,p) & (k+2,l,m-2,p)\\
(k+2,l,m,p-2) & (k+2,l+1,m-1,p-2)\\
\delta(k+2,l+1) &
\end{array}$$

where $\delta = 1$ *if* $m = 2$ *and* $p = 1$ *and* $\delta = 0$ *otherwise*.

(iii) *If* $\lambda \equiv (0,0,1,0) \pmod 2$ *or* $\lambda \equiv (1,1,0,1) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2*-regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k+1,l-1,m,p)\\
(k+1,l,m-1,p) & (k+1,l,m,p-1)\\
(k+1,l,m+1,p-2) & (k+1,l+1,m-2,p)\\
(k+1,l+1,m,p-2) & (k+1,l+2,m-1,p-2)\\
\delta(k+1,l+2) &
\end{array}$$

where $\delta = 1$ *if* $m = 2$ *and* $p = 1$ *and* $\delta = 0$ *otherwise*.

(iv) *If* $\lambda \equiv (0,0,1,1) \pmod 2$ *or* $\lambda \equiv (1,1,0,0) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2-*regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k,l,m+1,p-1) \\
(k,l+1,m,p-1) & (k,l+2,m-1,p-1) \\
(k+1,l-1,m,p) & (k+1,l-1,m+1,p-1) \\
(k+1,l,m-1,p) & (k+1,l+1,m-2,p) \\
(k+1,l+1,m-1,p-1) & 2(k+1,l+1,m,p-2) \\
2(k+2,l,m-1,p-1) & (k+2,l,m,p-2) \\
(k+2,l,m+1,p-3) & (k+2,l+2,m-1,p-3) \\
(k+3,l-1,m,p-2) & (k+3,l+1,m-2,p-2)
\end{array}$$

(v) *If* $\lambda \equiv (0,1,0,0) \pmod 2$ *or* $\lambda \equiv (1,0,1,1) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2-*regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k,l,m+1,p-1) \\
(k,l+1,m,p-1) & (k,l+2,m-1,p-1) \\
(k+1,l,m,p-1) & (k+2,l-1,m,p-1) \\
(k+2,l,m-1,p-1) & (k+2,l+1,m-2,p-1)
\end{array}$$

(vi) *If* $\lambda \equiv (0,1,0,1) \pmod 2$ *or* $\lambda \equiv (1,0,1,0) \pmod 2$ *then* $\mathscr{S}(k,l,m,p) = \mathscr{D}(k,l,m,p)$

(vii) *If* $\lambda \equiv (0,1,1,0) \pmod 2$ *or* $\lambda \equiv (1,0,0,1) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following* (2-*regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll}
(k,l,m,p) & (k,l+1,m-1,p) \\
(k,l+1,m,p-1) & (k,l+1,m+1,p-2) \\
(k+1,l,m-1,p) & 2(k+1,l,m,p-1) \\
(k+1,l,m+1,p-2) & (k+1,l+1,m-1,p-1) \\
(k+1,l+2,m-2,p-1) & (k+1,l+2,m,p-3) \\
(k+2,l-1,m-1,p) & (k+2,l-1,m,p-1) \\
(k+2,l-1,m+1,p-2) & 2(k+2,l+1,m-1,p-2) \\
(k+3,l,m-2,p-1) & (k+3,l,m,p-3).
\end{array}$$

(viii) *If* $\lambda \equiv (0,1,1,1) \pmod 2$ *or* $\lambda \equiv (1,0,0,0) \pmod 2$ *then the simple modules contained in* $\mathscr{S}(k,l,m,p)$ *are indexed by the following*

The Principal Block of $\mathscr{H}_{20}$

20	1
19,1	1	1
18,2	.	1	1
17,3	.	.	1	1
16,4	.	.	.	1	1
15,5	.	.	.	.	1	1
14,6	.	.	.	.	.	1	1
13,7	.	.	.	.	.	.	1	1
12,8	.	.	.	.	.	.	.	1	1
11,9	.	.	.	.	.	.	.	.	1	1
16,3,1	.	.	1	1	1	.	.	.	.	.	1
14,5,1	.	.	.	.	1	1	1	.	.	.	.	1
12,7,1	.	.	.	.	.	.	1	1	1	.	.	.	1
10,9,1	.	.	.	.	.	.	.	.	1	1	.	.	.	1
15,3,2	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1
14,4,2	.	.	.	.	1	.	.	.	.	.	.	1	.	.	1	1
13,5,2	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	1	1
12,6,2	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	1
11,7,2	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1	1
10,8,2	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	1	1
12,5,3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	1
10,7,3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	1
11,5,4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1
10,6,4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	1
9,7,4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	1
8,7,5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1
13,4,2,1	.	.	.	.	1	1	1	.	.	.	.	1	.	.	1	2	1	.	.	.	.	.	.	.	.	.	1
11,6,2,1	.	.	.	.	.	.	1	1	1	.	.	.	1	.	.	.	1	2	1	.	.	.	.	.	.	.	.	1
9,8,2,1	.	.	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	1	2	.	.	.	.	.	.	.	.	1
12,4,3,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	.	.	1	.	.	.	.	.	1	.	.	1
11,5,3,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	2	.	.	1	.	1	.	.	.	.	1	.	1	1
10,6,3,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	.	1	1	1	.	.	.	1	.	.	1	1
9,7,3,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	2	.	1	.	1	1	.	.	.	1	.	.	1	1
9,6,4,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	1	2	1	.	.	.	.	.	1	1	1	1
8,6,5,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	.	.	.	.	.	.	1	1	1
10,5,3,2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	1	1	.	.	.	1	.	1	2	1	.	.	.	1
8,7,3,2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	1	1	.	.	.	1	.	.	1	2	.	.	.	1
9,5,4,2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	2	.	.	.	.	.	.	2	1	1	1	.	1	.	1
8,6,4,2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	2	1	.	.	.	.	.	1	1	2	1	1	.	1	1	1
7,6,4,3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	2	.	1	.	1	1	2	1
8,5,4,2,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	1	5	1	1	.	.	.	.	2	1	2	1	1	1	1	2	2	.	1
7,5,4,3,1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	1	.	.	.	.	.	.	2	.	1	.	1	2	4	1	1	1

SCHEME 1

(2-*regular*) *partitions with the stated multiplicities*:

$$\begin{array}{ll} (k, l, m, p) & (k, l, m+1, p-1) \\ (k, l+1, m-1, p) & (k, l+2, m, p-2) \\ (k+1, l, m-1, p) & (k+2, l-1, m-1, p) \\ (k+2, l, m, p-2) & (k+2, l+1, m-1, p-2) \end{array}$$

Proof. This is a simple exercise using Theorem 8.1 so we only outline what needs to be done. First note that by Theorem 7.1 we know exactly which four-part partitions contain simple modules indexed by partitions of length 2; this accounts for the two instances of δ above (note that simples indexed by 2-part partitions may also appear in (vii); however, here they are "generic"). If $\mathscr{D}(\mu) \subseteq \mathscr{S}(\lambda)$ and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ then by using Theorem 1.2 we can remove the first μ_4 columns from λ and μ without changing the decomposition number $d_{\lambda\mu}$. This reduces the calculation to deciding when a Specht module indexed by a partition of length 3 or 4 appears in an indecomposable module indexed by a 3-part partition. It is now a straightforward exercise to deduce the corollary from the lists given in the results from (8.6) through to Theorem 8.18. ∎

APPENDIX: THE PRINCIPAL BLOCK OF THE HECKE ALGEBRA $\mathscr{H}_{20}$

Using the results of this paper and the remark at the end of Section 4 is it straightforward to calculate the decomposition matrices of $\mathscr{H}_n$ for $n \leq 19$ (for $n \leq 10$ these are given in [8]). In calculating the full decomposition matrix for $\mathscr{H}_{20}$ the only real difficulties are in deciding $d_{(7,5,4,3,1)(8,6,4,2)}$ and $d_{(8,5,4,2,1)(10,6,4)}$. It was only recently, using the main result of [10], that we were able to calculate these two decomposition numbers. In Scheme 1 we reproduce the 2-regular part of the principal block of the decomposition matrix of $\mathscr{H}_{20}$.

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